Public Key Cryptography

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1

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Outline

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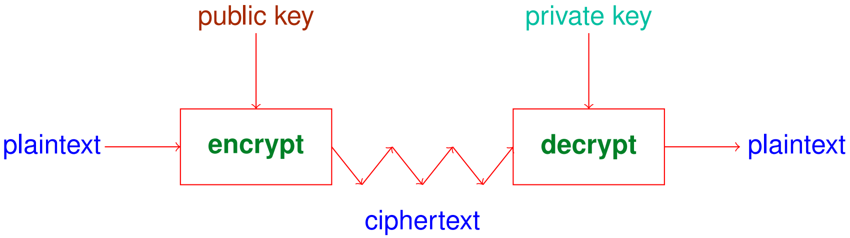
A Generic View of Public Key Crypto



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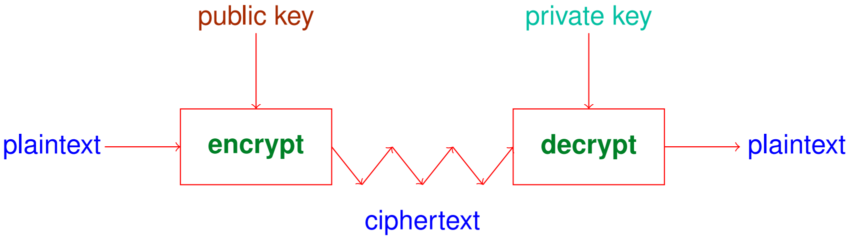

Advantages over symmetric-key

1. Better key distribution and management
   - No danger that public key compromised
2. New protocols
   - Digital Signature
3. Long-term encryption

Only disadvantage:

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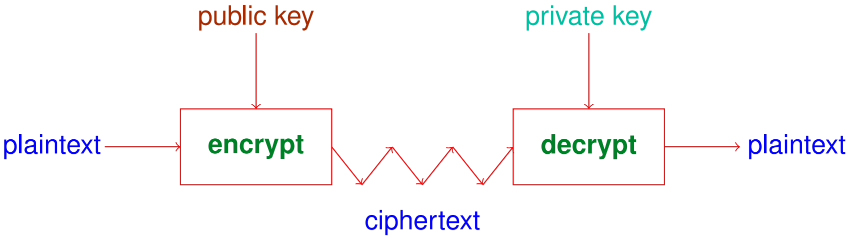

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1. Better key distribution and management
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   - Digital Signature
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Only disadvantage: much more slower than symmetric key crypto

Definition

PKC

A public key cryptosystem is a pair of families $\{E_k : k \in \mathcal{K}\}$ and $\{D_k : k \in \mathcal{K}\}$ of algorithms representing invertible transformations,

$$E_k : \mathcal{M} \to \mathcal{C} \;\&\; D_k : \mathcal{C} \to \mathcal{M}$$

on a finite message space $\mathcal{M}$ and ciphertext space $\mathcal{C}$, such that

(i) for every $k \in \mathcal{K}$, D_k is the inverse of E_k and vice versa,

(ii) for every $k \in \mathcal{K}$, $M \in \mathcal{M}$ and $C \in \mathcal{C}$, the algorithms E_k and D_k are *easy* to compute.

(iii) for every $k \in \mathcal{K}$, it is feasible to compute inverse pairs E_k and D_k from k,

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(iii) for every $k \in \mathcal{K}$, it is feasible to compute inverse pairs E_k and D_k from k,

(iv) for almost every $k \in \mathcal{K}$, each easily computed algorithm equivalent to D_k is computationally infeasible to derive from E_k, without knowing k.

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Computationally Infeasible

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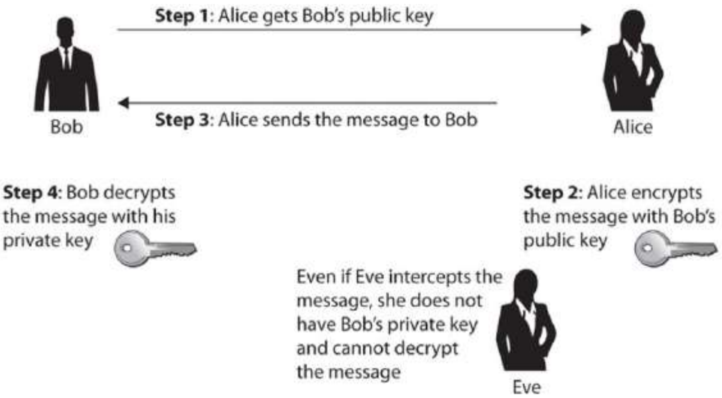
Any computational task which takes $\geq 2^{112}$ bit operations, we say, it is computationally infeasible in present day scenario.

PKC


Step 1: Alice gets Bob's public key
Bob
Alice
Step 3: Alice sends the message to Bob
Step 4: Bob decrypts the message with his private key
Step 2: Alice encrypts the message with Bob's public key
Even if Eve intercepts the message, she does not have Bob's private key and cannot decrypt the message
Eve

Digital Signature

$\boxed{\textbf{Signing a Message } M}$

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$$\boxed{\text{Message } M} \xrightarrow{\text{Hash Function } h} \boxed{\text{Digest } h(M)}$$

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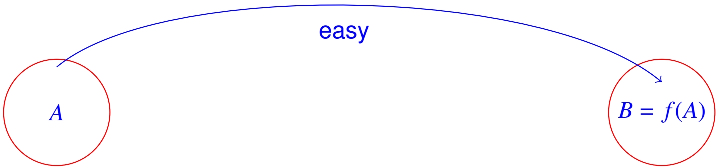
One-way Function



Definition

Easy: $\exists$ *a polynomial-time algorithm that, on input* $m \in A$ *outputs* $c = f(m)$.

Definition

Hard: *Every probabilistic polynomial-time algorithm trying, on input* $c (= f(m))$ *to find an inverse of* $c \in B$ *under* f, *may succeed only with negligible probability.*

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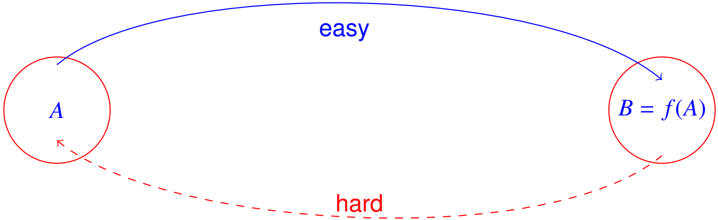

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- The function
$$f : \mathbb{Z}_p \to \mathbb{Z}_p,$$
$$x \mapsto x^{2^{24}+17} + a_1.x^{2^{24}+3} + a_2.x^3 + a_3.x^2 + a_4.x + a_5,$$
where $p = 2^{64} - 59$ and each $a_i\ (\in \mathbb{Z}_p)$ is 19-digit number for $1 \le i \le 5$.

Trapdoor One-way Function

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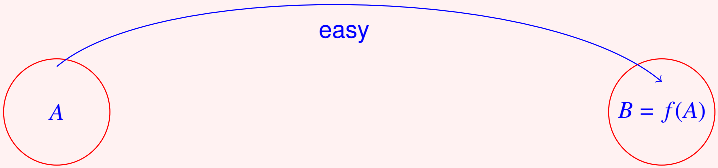

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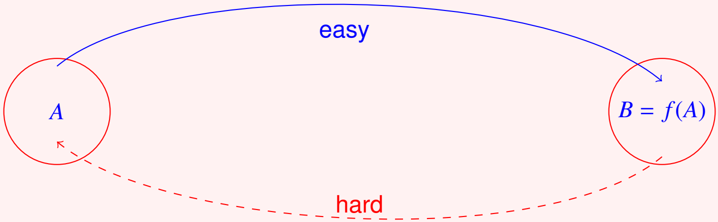

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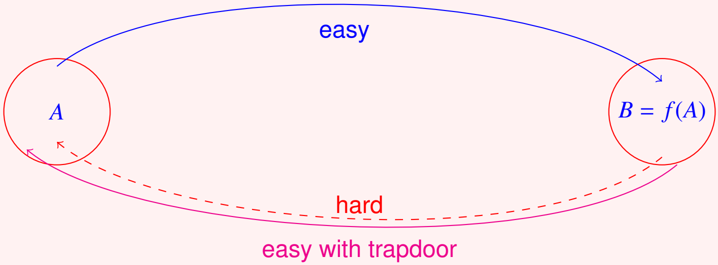

Trapdoor One-way Function

Definition

A trapdoor one-way function is a one-way function $f : \mathcal{M} \rightarrow \mathcal{C}$, satisfying the additional property that $\exists$ some additional information or trapdoor that makes it easy for a given $c \in f(\mathcal{M})$ to find out $m \in \mathcal{M} : f(m) = c$, but without the trapdoor this task becomes hard.

Examples Trapdoor One-way Function

- **Integer Factorization:** Given $n \in \mathbb{Z}^+$, find $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ where the p_i are pairwise distinct primes and each $e_i \geq 0$ for $1 \leq i \leq k$. $\rightarrow$ **hard problem**.

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Example

- Consider the number 37015031

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- **Discrete Logarithm Problem:** Given an abelian group $(G, \ .)$ and $g \in G$ of order n. Given $h \in G$ such that $h = g^x$ find x ($DLP(g, h) \rightarrow x$). $\rightarrow$ **hard problem**.

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Example

- Let $p = 97$. Then $\mathbb{Z}_{97}^*$ is a cyclic group of order $n = 96$.
  5 is a generator of $\mathbb{Z}_{97}^*$.
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- **Computational Diffie-Hellman Problem:** Given $a = g^x$ and $b = g^y$ find $c = g^{xy}$. ($CDH(g, a, b) \rightarrow c$). $\rightarrow$ **hard problem**.
- **Elliptic Curve Discrete Logarithm Problem (ECDLP):** $\mathbb{E}$ denotes the collections of points on a elliptic curve and $P \in \mathbb{E}$. Let $\mathcal{S}$ be the cyclic subgroup of $\mathbb{E}$ generated by P. Given $Q \in \mathcal{S}$, find an integer x such that $Q = x.P$. $\rightarrow$ **hard problem**.

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DH Key Exchange

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- PKCS #3 (Version 1.4): Diffie-Hellman Key-Agreement Standard, An RSA Laboratories Technical Note – Revised November 1, 1993.

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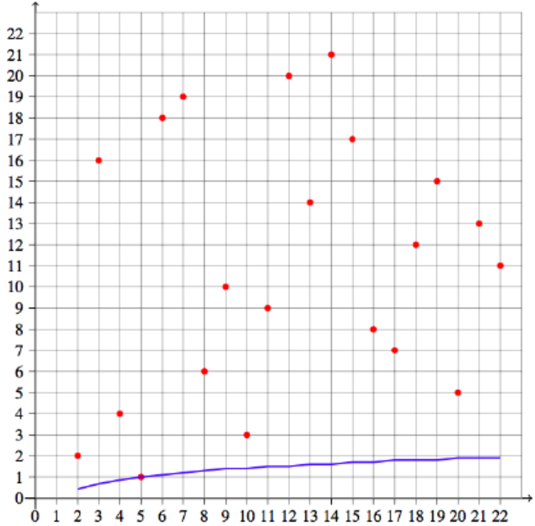

- Clifford Cocks, Malcolm Williamson & James Ellis developed Nonsecret Encryption between 1969 and 1974.

Clifford Cocks, Malcolm Williamson, and James Ellis.

- All were at GCHQ, so this stayed secret until 1997.

Nonsecret Encryption

Key Generation

1. Select 2 large distinct primes p & q such that $p \nmid (q-1)$ and $q \nmid (p-1)$.

   Public key: $n = pq$.

2. Find numbers r & s, s/t $p.r \equiv 1 \mod (q-1)$ and $q.s \equiv 1 \mod (p-1)$.
3. Find u & v, s/t $u.p \equiv 1 \mod q$ and $v.q \equiv 1 \mod p$.

   Private key: (p, q, r, s, u, v).

Nonsecret Encryption

Encryption

$$C \equiv M^n \mod n \text{ for } 0 \leq M < n.$$

Decryption

1. $a \equiv C^s \mod p$ and $b \equiv C^r \mod q$.
2. $M \equiv a.q.v + b.p.u \mod n$.

Outline

RSA Key Generation

- Generate two large distinct random primes p & q.
- Compute $n = pq$ and $\phi(n) = (p-1)(q-1)$.
- Select a random integer $e,\ 1 < e < \phi(n)$ s/t $\gcd(e, \phi(n)) = 1$.

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- Compute the unique integer $d,\ 1 < d < \phi(n)$ s/t

$$ed \equiv 1 \mod \phi(n).$$

Public key is (n, e); Private key is (p, q, d).

RSA Encryption/Decryption

Encryption:

$$c \equiv m^e \mod n,$$

Plaintext m and ciphertext $c \in \mathbb{Z}_n$.

Decryption:

$$m' \equiv c^d \mod n.$$

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PKCS #1 v2.2: RSA Cryptography Standard, RSA Laboratories – October 27, 2012.

RSA Validation

SBI Public Key Information

Public Key Info

Algorithm	RSA
Key Size	2048
Exponent	65537
Modulus	A6:55:7F:B2:9C:23:FC:79:F8:9D:90:F6:75:4E:CE:3A:26:90:B8:37:EA:8E:6E: D6:18:8A:FC:F6:CA:7C:6F:4B:45:4D:98:DE:4F:3D:A3:78:5E:0C:4A:1A:81:8D: 6F:C3:BB:4C:38:6E:04:0B:1F:BB:CB:50:8B:42:E9:E2:17:65:E2:C0:D0:CA:F4: E5:C6:0A:C9:47:53:32:15:69:F6:C4:EC:B0:E0:B0:FC:CB:BA:DE:DF:BE:ED:2 B:44:3D:F6:2B:B3:0A:CA:B8:FC:D1:5F:84:2C:34:1E:15:52:76:4E:90:FA:85:7 0:BB:05:C3:02:03:17:74:B3:80:A1:59:1F:19:7B:3A:2B:C3:D5:59:CF:BA:5D:B E:DF:3B:3A:8E:52:C1:D3:A3:8C:06:D2:2A:98:2F:4D:82:7F:28:F1:B1:D3:71:7 E:CF:4C:B1:26:F4:6F:EA:09:F9:7F:5A:D6:15:46:5C:92:50:D4:F4:F3:CA:60:2 5:4D:9A:66:91:1D:EA:74:D4:B1:71:D9:30:15:4C:BB:B6:CD:C6:18:82:F8:B7:4 8:97:AF:2F:22:15:94:FE:EB:E7:DE:EF:CA:A3:6E:CC:26:69:D5:92:5B:68:89:5 6:2B:B3:72:60:62:49:8B:C5:59:45:43:C1:F4:7E:8F:2B:C4:DD:C1:BB:39:D4:B C:5C:51:53

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Definition

A prime p *is called a strong prime if*

Strong Prime Number

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(i) $p-1$ *has a large prime factor, say* r*,*

(ii) $p+1$ *has a large prime factor, and*

(iii) $r-1$ *has a large prime factor.*

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ii. The Euler phi function is multiplicative. That is, if $gcd(m, n) = 1$, then

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iii. If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, is the prime factorization of n, then

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

Modular Arithmetic

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- **Euler's theorem:** If $a \in \mathbb{Z}_n^*$, then

$$a^{\phi(n)} \;\equiv\; 1 \; mod \; n.$$

Pseudoprime

Definition

*If n is an odd composite number and b is an integer s/t $\gcd(n,b)=1$ and $b^{n-1}\equiv 1 \mod n$ then n is called a **pseudoprime** to the base b. The integer b is called a **Fermat liar** (to primality) for n.*

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Example

1. The number $n = 91$ is a pseudoprime to the base $b = 3$,

$$\because 3^{90} \equiv 1 \mod 91.$$

2. However, 91 is not a pseudoprime to the base 2,

$$\because 2^{90} \equiv$$

3. The composite integer $n = 341 (= 11 \times 31)$ is a pseudoprime to the base $2, \because 2^{340} \equiv 1 \mod 341$.

Carmichael Number

Definition

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1. $n = 561 = 3 \times 11 \times 17$ is a Carmichael number. This is the smallest Carmichael number.
2. The following are Carmichael numbers:
   (a) $1105 = 5 \times 13 \times 17$
   (b) $1729 = 7 \times 13 \times 19$
   (c) $2465 = 5 \times 17 \times 29$

Carmichael Number

- A composite integer n is a Carmichael number iff the following two conditions are satisfied:
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- A Carmichael number must be the product of at least three distinct primes.
- There are an infinite number of Carmichael numbers.

Quadratic Residue

Definition

*Let $a \in \mathbb{Z}_n^*$; a is said to be a **quadratic residue** modulo n, if*

$$\exists\, x \in \mathbb{Z}_n^* \ni x^2 \equiv a \mod n.$$

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$\alpha = 6$ is a generator of $\mathbb{Z}_{13}^*$. The powers of α are

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  Let $n = 21$.
  Then $Q_{21} = \{1, 4, 16\}$ and $\overline{Q_{21}} = \{2, 5, 8, 10, 11, 13, 17, 19, 20\}$.

The Legendre and Jacobi Symbols

- Let p be an odd prime and a an integer. The **Legendre symbol** $\left(\frac{a}{p}\right)$ is defined to be

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a, \\ 1, & \text{if } a \in Q_p, \\ -1, & \text{if } a \in \overline{Q_p}. \end{cases}$$

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- Let $n \geq 3$ be odd with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Then the **Jacobi symbol** $\left(\frac{a}{n}\right)$ is defined to be

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{e_1} \left(\frac{a}{p_2}\right)^{e_2} \cdots \left(\frac{a}{p_k}\right)^{e_k}$$

Properties of Legendre Symbol

(i) $\left(\frac{a}{p}\right) = a^{(p-1)/2} \mod p$. In particular, $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$. Hence, $-1 \in Q_p$ if $p \equiv 1 \mod 4$, and $-1 \in \overline{Q_p}$ if $p \equiv 3 \mod 4$.

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3. (iii) If $a \equiv b \mod p$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.
4. (iv) **Law of quadratic reciprocity:** If q is an odd prime distinct from p, then

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{(p-1)(q-1)/4}.$$

Fermat Test for Primality – Probabilistic Algorithm

Fermat Test for Primality

Input: n
Output: YES if n is composite, NO otherwise.
Choose a random $b,\ 0 < b < n$
if $\gcd(b, n) > 1$ **then**
  return YES
end
else ;
if $b^{n-1} \not\equiv 1 \mod n$ **then**
  return YES
end
else ;
return NO

The Euler Test – Probabilistic Algorithm

- If n is an odd prime, we know that an integer can have at most two square roots, $\mod n$. In particular, the only square roots of $1 \mod n$ are ± 1.
- If $a \not\equiv 0 \mod n$, $a^{(n-1)/2}$ is a square root of $a^{n-1} \equiv 1 \mod n$, so $a^{(n-1)/2} \equiv \pm 1 \mod n$.

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- If $a^{(n-1)/2} \not\equiv \pm 1 \mod n$ for some a with $a \not\equiv 0 \mod n$, then n is composite.

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- For a randomly chosen a with $a \not\equiv 0 \mod n$, compute $a^{(n-1)/2} \mod n$.

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  If n is large and chosen at random, the probability that n is prime is very close to 1.

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The Euler test is more powerful than the Fermat test.

The Euler Test – Probabilistic Algorithm

The Euler test is more powerful than the Fermat test.

- If the Fermat test finds that n is composite, so does the Euler test.
- If n is an odd composite integer (other than a prime power), 1 has at least 4 square roots $\mod n$.

  So we can have $a^{(n-1)/2} \equiv \beta \mod n$, where $\beta \neq \pm 1$ is a square root of 1.

  Then $a^{n-1} \equiv 1 \mod n$. In this situation, the Fermat Test (incorrectly) declares n a probable prime, but the Euler test (correctly) declares n composite.

Miller-Rabin Test – Probabilistic Algorithm

- The Euler test improves upon the Fermat test by taking advantage of the fact, if 1 has a square root other than $\pm 1 \mod n$, then n must be composite.
- If $a^{(n-1)/2} \not\equiv \pm 1 \mod n$, where $\gcd(a, n) = 1$, then n must be composite for one of two reasons:
  - (i) If $a^{n-1} \not\equiv 1 \mod n$, then n must be composite by Fermat's Little Theorem
  - (ii) If $a^{n-1} \equiv 1 \mod n$, then n must be composite because $a^{(n-1)/2}$ is a square root of $1 \mod n$ different from ± 1.

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  2. If $a^{n-1} \equiv 1 \mod n$, then n must be composite because $a^{(n-1)/2}$ is a square root of $1 \mod n$ different from ± 1.
- The limitation of the Euler test is that is does not go to any special effort to find square roots of 1, different from ± 1. The Miller-Rabin test does this.

Miller-Rabin Test – Probabilistic Algorithm

Miller-Rabin Test

Input: an odd integer $n \geq 3$ and security parameter $t \geq 1$.
Output: an answer "prime" or "composite" to the question: "Is n prime?"
Write $n - 1 = 2^s.r$ s/t r is odd.
for $i = 1$ *to* t **do**
  Choose a random integer a s/t $2 \leq a \leq n - 2$.
  Compute $y \equiv a^r \mod n$
  if $y \neq 1$ & $y \neq n - 1$ **then**
    $j \leftarrow 1$.
    while $j \leq s - 1$ & $y \neq n - 1$ **do**
      Compute $y \leftarrow y^2 \mod n$.
      If $y = 1$ then **return**("composite").
      $j \leftarrow j + 1$.
    end
    If $y \neq n - 1$ then **return** ("composite").
  end
end
Return("prime").

Deterministic Polynomial Time Algorithm

The AKS Algorithm

Input: a positive integer $n > 1$
Output: n is **Prime** or **Composite** in deterministic polynomial-time
If $n = a^b$ with $a \in \mathbb{N}$ & $b > 1$, then output **COMPOSITE.**

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If $n = a^b$ with $a \in \mathbb{N}$ & $b > 1$, then output **COMPOSITE.**
Find the smallest r such that $ord_r(n) > 4(\log n)^2$.
If $1 < \gcd(a, n) < n$ for some $a \leq r$, then output **COMPOSITE**.

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Find the smallest r such that $ord_r(n) > 4(\log n)^2$.
If $1 < \gcd(a, n) < n$ for some $a \leq r$, then output **COMPOSITE**.
If $n \leq r$, then output **PRIME.**
for $a = 1$ *to* $\lfloor 2\sqrt{\phi(r)}\log n \rfloor$ **do**
    if $(x - a)^n \not\equiv (x^n - a) \mod (x^r - 1, n)$,
    then output **COMPOSITE.**
end
Return("PRIME").

RSA Example

- Suppose A wants to send the following message to B

  RSAISTHEKEYTOPUBLICKEYCRYPTOGRAPHY

- B chooses his $n = 737 = 11 \times 67$. Then $\phi(n) = 660$. Suppose he picks $e = 7, \Rightarrow d = 283$.

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- $\because \; 26^2 < n < 26^3 \quad \therefore$ the block size of the plaintext $= 2$.

$$m_1 = \text{‘}RS\text{'} = 17 \times 26 + 18 = 460$$

$$c_1 = 460^7 \equiv 697 \mod 737 = 1.26^2 + 0.26 + 21 = BAV$$

RSA Example

	RS	AI	ST	HE	KE	YT	OP	UB
m_b	460	8	487	186	264	643	379	521
c_b	697	387	229	340	165	223	586	5

LI	CK	EY	CR	YP	TO	GR	AP	HY
294	62	128	69	639	508	173	15	206
189	600	325	262	100	689	354	665	673

RSA Example

- Suppose A wants to send the following message to B

power

- B chooses his $n = 1943 = 29 \times 67$. Then $\phi(n) = 1848$. Suppose he picks $e = 701, \Rightarrow d = 29$.
- $\because\ 26^2 < n < 26^3 \quad \therefore$ the block size of the plaintext $= 2$.
- $m_1 = \text{`}po' = 15 \times 26 + 14 = 404,\ m_2 = \text{`}we' = 22 \times 26 + 4 = 576,\ m_3 = \text{`}ra' = 17 \times 26 + 0 = 442$.
- $c_1 = 404^{701} \equiv 1419 \mod 1943 = 2.26^2 + 2.26 + 15 = ccp$.
- $\|ly,\ c_2 = 344 = 13.26 + 6 = ang$ & $c_3 = 210 = 8.26 + 2 = aic$.
- The cipher text is

ccpangaic

Security of RSA

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If we know n and $\phi(n)$, we can find p & q.

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We have

$$\phi(n) = pq - p - q + 1 = n - (p + q) + 1.$$

Since we know n, we can find $p + q$ from the above equation. Since we know $pq = n$ and $p + q$, we can find p & q by factoring the quadratic equation

$$x^2 - (p + q)x + pq = 0.$$

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- Security of RSA relies on difficulty of finding d given n & e.
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- Security of RSA relies on difficulty of finding d given n & e.
- Breaking RSA is no harder than Factoring.
- It is not secure against chosen ciphertext attacks (CCA).
  - **Input challenge** ciphertext $c \equiv m^e \mod N$.
  - Submit ciphertext $c' \equiv r^e c \mod N$ for decryption.
  - Receive message $m' = rm$.
  - Original message is $r^{-1}m' \mod N \equiv m$.
- RSA is secure against chosen plaintext attack (CPA).

IND-CCA

Security notion for encryption.

- From a ciphertext c, an attacker should not be able to derive any information from the corresponding plaintext m.
- Even if the attacker can obtain the decryption of any ciphertext, c excepted.
- This is called indistinguishability against a chosen ciphertext attack (IND-CCA).

SBI Public Key Information

Public Key Info

Algorithm	RSA
Key Size	2048
Exponent	65537
Modulus	A6:55:7F:B2:9C:23:FC:79:F8:9D:90:F6:75:4E:CE:3A:26:90:B8:37:EA:8E:6E: D6:18:8A:FC:F6:CA:7C:6F:4B:45:4D:98:DE:4F:3D:A3:78:5E:0C:4A:1A:81:8D: 6F:C3:BB:4C:38:6E:04:0B:1F:BB:CB:50:8B:42:E9:E2:17:65:E2:C0:D0:CA:F4: E5:C6:0A:C9:47:53:32:15:69:F6:C4:EC:B0:E0:B0:FC:CB:BA:DE:DF:BE:ED:2 B:44:3D:F6:2B:B3:0A:CA:B8:FC:D1:5F:84:2C:34:1E:15:52:76:4E:90:FA:85:7 0:BB:05:C3:02:03:17:74:B3:80:A1:59:1F:19:7B:3A:2B:C3:D5:59:CF:BA:5D:B E:DF:3B:3A:8E:52:C1:D3:A3:8C:06:D2:2A:98:2F:4D:82:7F:28:F1:B1:D3:71:7 E:CF:4C:B1:26:F4:6F:EA:09:F9:7F:5A:D6:15:46:5C:92:50:D4:F4:F3:CA:60:2 5:4D:9A:66:91:1D:EA:74:D4:B1:71:D9:30:15:4C:BB:B6:CD:C6:18:82:F8:B7:4 8:97:AF:2F:22:15:94:FE:EB:E7:DE:EF:CA:A3:6E:CC:26:69:D5:92:5B:68:89:5 6:2B:B3:72:60:62:49:8B:C5:59:45:43:C1:F4:7E:8F:2B:C4:DD:C1:BB:39:D4:B C:5C:51:53

LinkedIn Public Key Information

Public Key Info

Algorithm	RSA
Key Size	2048
Exponent	65537
Modulus	D4:8A:8B:DF:28:F5:5C:7B:B6:79:74:E5:F4:4A:5B:E7:38:94:69:B7:BA:19:4D: A7:A9:73:64:6F:DD:B8:4C:99:5A:91:E8:F5:C8:D7:B1:1E:5B:3E:3E:AE:77:6B: A3:E3:DF:D3:29:38:59:E8:66:59:5D:37:FF:75:20:4E:66:1B:D0:C8:73:9E:A0: 38:6E:16:98:BD:DB:CC:D8:95:CF:87:AE:5E:42:10:F8:10:34:BF:E8:1F:5A:0A: 4B:A3:28:25:55:3F:FD:15:D0:3D:25:EF:09:6C:E4:C0:E4:9F:E7:4E:28:C6:D0: 63:2C:07:4C:CE:4F:4E:EE:B1:70:66:07:96:40:E3:51:1B:23:91:84:12:AE:A5:F A:2D:B0:3E:1E:C1:AC:BF:80:90:31:81:88:C7:5C:66:0E:34:5F:62:B5:CF:03:8 E:C8:74:82:77:01:A1:E8:A1:D3:1D:4B:43:6A:87:F2:E2:22:48:58:B2:3A:88:C7: F8:DC:9D:70:D9:BE:83:E1:B2:E9:BA:AC:C5:EF:B0:CB:76:9D:6E:10:F7:C9:80: 6E:B7:C7:30:5B:85:5F:D9:6C:26:B1:B9:59:24:17:C5:F6:01:CD:67:FA:21:E8:B B:1D:24:44:20:6B:09:CA:8F:5B:10:AF:76:B0:AB:33:9F:28:B2:B1:C8:FC:2F:E 5:71

IIITL Public Key Information

Public Key Info

Algorithm	RSA
Key Size	2048
Exponent	65537
Modulus	BF:26:C8:BA:E3:2F:68:5A:8F:C1:82:43:AC:0A:82:B5:0D:4E:04:6E:B1:85:35: 8E:14:51:AC:7A:44:4F:A5:CF:A2:3C:4C:8B:97:7E:0E:8C:4A:F6:05:1F:53:5C:4 E:D1:1D:23:84:8C:8F:C7:B6:99:AA:6D:00:36:E4:FF:53:7F:EC:FF:9F:42:B9:2 B:F5:EF:39:9B:7C:F3:51:75:0F:0C:B1:AA:FB:4C:59:40:06:C5:60:0F:5D:2F:A 8:47:CE:47:CF:69:73:0B:AB:71:44:51:01:6D:E1:C8:9A:EF:FA:96:A4:E7:AF:5E: 1F:4B:A7:6C:26:8A:7B:4E:A9:14:7A:EC:74:7B:7B:D3:9B:51:C7:60:1F:E7:CB:7 F:E9:A8:F2:C5:6F:22:4A:42:AB:60:B5:BF:D9:9D:CA:D7:6D:F2:8C:06:6E:30: A5:F1:AB:EC:32:73:D3:E8:67:93:E3:06:C9:58:C5:99:43:8C:5E:3C:C2:7A:B9: 1B:27:47:29:B7:9E:9A:DC:FB:63:6A:E0:A1:BC:33:B0:FE:C1:12:6F:01:73:A7:A B:3E:C9:92:EB:45:FE:5D:86:CA:4D:99:87:6E:75:4C:B3:CD:85:F0:AE:61:9B:B C:C6:9E:A4:3A:D2:53:76:EE:73:D9:3A:52:0C:CD:D1:73:70:7A:D5:BC:DC:5E: 58:7D

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- If adversary gets hold of the messages $y_i, \ 1 \leq i \leq 3$, (s)he can compute $M^3 \mod n_1 n_2 n_3$ using Chinese remainder theorem since $\gcd(n_i, n_j) = 1$ for $i \neq j$.

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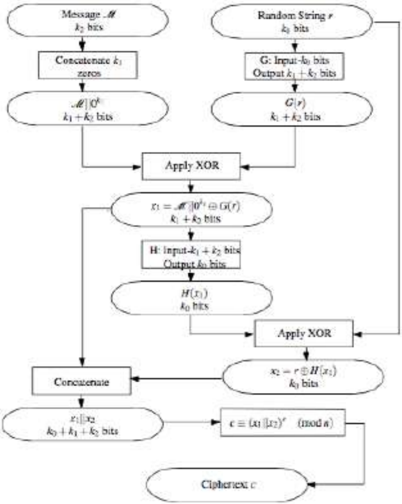
- $\because \ m < n_i, \ m^3 < n_1 n_2 n_3$. So, $M^3 \mod n_1 n_2 n_3 = M^3$ and the adversary can find M by taking the cube root of $M^3 \mod n_1 n_2 n_3$.

RSA in Practice – Optimal Asymmetric Encryption Padding (OAEP)



Optimal Asymmetric Encryption Padding (OAEP) I

- To encrypt a message M of k_2-bit, first concatenates the message with 0^{k_1}.
- Expands the message to $M||0^{k_1}$.
- After that, select a random string r of length k_0 bits.
- Use it as the random seed for $G(r)$ and computes

$$x_1 = (M||0^{k_1}) \oplus G(r), \quad x_2 = r \oplus H(x_1)$$

- If $x_1||x_2$ is a binary number bigger than n, Alice chooses another random string r and computes the new values of x_1 & x_2.
- If $G(r)$ produces fairly random outputs, $x_1||x_2$ will be less than n in binary with a probability greater than $\frac{1}{2}$.

Optimal Asymmetric Encryption Padding (OAEP) II

- After getting a string r with $x_1 \| x_2 < n$, Alice then encrypts $x_1 \| x_2$ to get the ciphertext

$$E(M) = (x_1 \| x_2)^e \equiv c \mod n$$

ElGamal PKC in $\mathbb{Z}_p^*$

This was designed by Taher ElGamal in 1985

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Key Generation:

- $<\alpha>= \mathbb{Z}_p^*,\ \mathcal{P} = \mathbb{Z}_p^*\ \&\ \mathcal{C} = \mathbb{Z}_p^* \times \mathbb{Z}_p^*$.
- $\beta \equiv \alpha^a \mod p$.
- Public: p, α, β and Private: a.

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Encryption:

- Select a random $k \in \mathbb{Z}_{p-1}$.
- $Enc_k(x) = (y_1, y_2)$

$$y_1 \equiv \alpha^k \mod p, \quad y_2 \equiv x.\beta^k \mod p.$$

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Decryption:

$$Dec_k(y_1, y_2) \equiv y_2.(y_1^a)^{-1} \mod p.$$

ElGamal PKC in $\mathbb{Z}_p^*$

Example

- Let $p = 29$ and $\alpha = 2$, α is a primitive element $\mod 29$.
- Let $a = 5$,

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- **Public Key:** $(29, 2, 3)$ and **Private Key:** 5
- **Plaintext:** $x = 6$ & random number $k = 14 \in \mathbb{Z}_{28}$

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Example

- Let $p = 29$ and $\alpha = 2$, α is a primitive element $\mod 29$.
- Let $a = 5$, $\therefore \beta \equiv 2^5 \mod \equiv 3 \mod 29$.
- **Public Key:** $(29, 2, 3)$ and **Private Key:** 5
- **Plaintext:** $x = 6$ & random number $k = 14 \in \mathbb{Z}_{28}$
- $$y_1 \equiv 2^{14} \equiv 28 \mod 29 \;\&\; y_2 \equiv 6.3^{14} \equiv 23 \mod 29$$
- **Ciphertext:** $(28, 23)$.

Security of ElGamal Ciphertexts

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- Now suppose instead that Eve claims to possess the message m corresponding to an ElGamal encryption (r, t).
- Can you verify her claim?
- This is as hard as the decision Diffie-Hellman problem.

Elliptic Curves

- Elliptic curve[1] E over field $\mathbb{K}$ is defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{K}$$

- The set of $\mathbb{K}$-rational points $E(\mathbb{K})$ is defined as

$$E(\mathbb{K}) = \{(x, y) \in \mathbb{K} \times \mathbb{K} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{\mathcal{O}\}$$

[1] It is called a (generalized) Weierstrass equation. The equation defines a cubic curve called a Weierstrass curve.

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Theorem

There exists an addition law on E and the set $E(K)$ with that addition forms a group.

[1] It is called a (generalized) Weierstrass equation. The equation defines a cubic curve called a Weierstrass curve.

Elliptic Curves

1. Let $\mathbb{K}$ be a field of characteristic $\neq 2, 3$, and let $x^3 + ax + b$ be a cubic polynomial with no multiple roots, i.e., when

$$-16(4a^3 + 27b^2) \neq 0 \Rightarrow 4a^3 + 27b^2 \neq 0.$$

An elliptic curve over $\mathbb{K}$ is the set of points (x, y) with $x, y \in K$ which satisfy the equation

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2. If char $K = 2$, then an elliptic curve over $\mathbb{K}$ is the set of points satisfying an equation of type either

$$y^2 + cy = x^3 + ax + b \text{ or } y^2 + xy = x^3 + ax + b$$

   together with the *point at infinity* O.

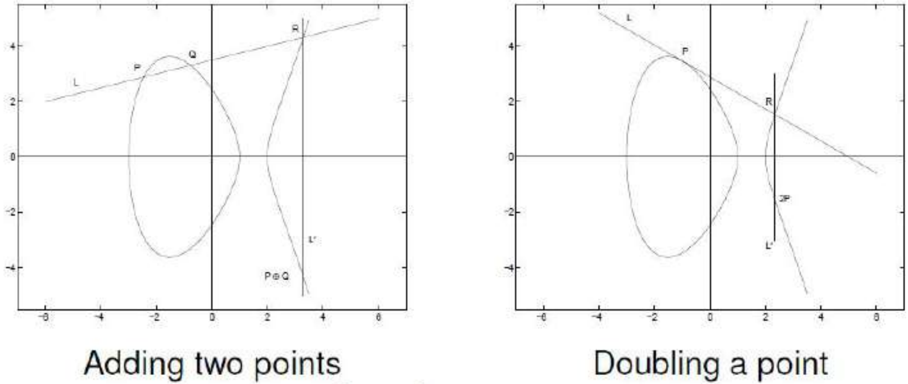
Elliptic Curves

3. If char $K = 3$, then an elliptic curve over $\mathbb{K}$ is the set of points satisfying the equation

$$y^2 = x^3 + ax^2 + bx + c$$

together with the *point at infinity* O.

Addition Law on Elliptic Curves



$$y^2 = x^3 - 7x + 6$$

Addition Law on Elliptic Curves I

- Suppose E is a nonsingular elliptic curve.
- The point at infinity $\mathcal{O}$, will be the identity element, so $P + \mathcal{O} = \mathcal{O} + P = P \; \forall \; P \in E$.
- Suppose $P, Q \in E$, where $P = (x_1, y_1)$ & $Q = (x_2, y_2)$

  (i) $x_1 \neq x_2$

  - L is the line through P and Q.
  - L intersects E in the two points P and Q
  - L will intersect E in one further point R'.
  - If we reflect R' in the x-axis, then we get a point R.

$$P + Q = R.$$

Addition Law on Elliptic Curves II

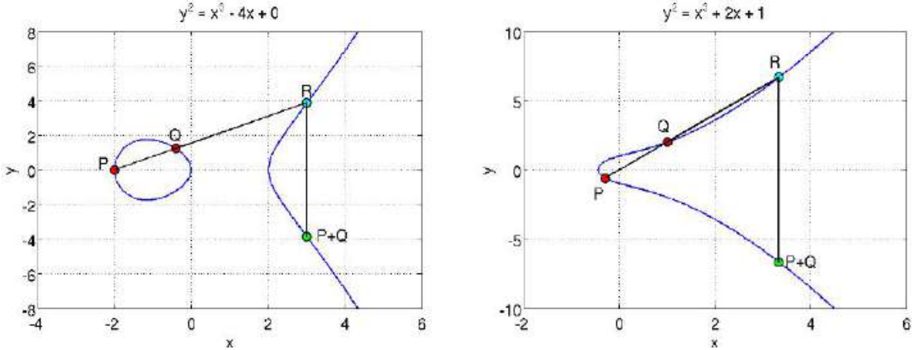
(ii) $x_1 = x_2 \;\&\; y_1 = -y_2$

$$(x, y) + (x, -y) = \mathcal{O}$$

(iii) $x_1 = x_2 \;\&\; y_1 = y_2$

- Draw a tangent line L through P
- Follow step (i)

Addition Law on Elliptic Curves



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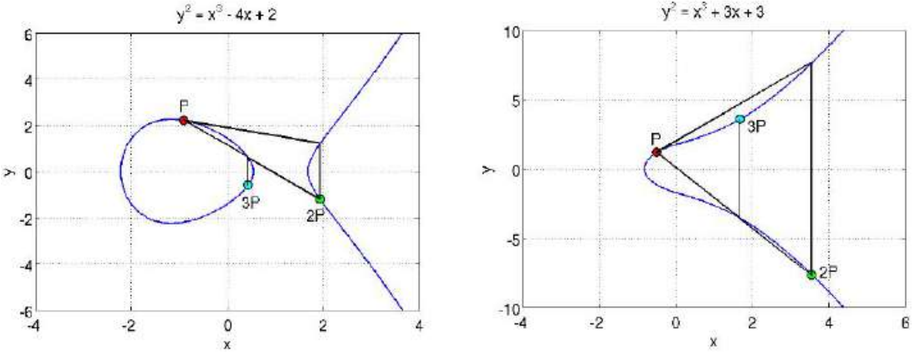

Addition Law on Elliptic Curves

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$$\lambda = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\ \frac{3x_1^2 + a}{2y_1} & \text{if } P_1 = P_2 \end{cases} \qquad \text{and} \qquad \nu = y_1 - \lambda x_1$$

Addition Law on Elliptic Curves

- Thus, we have

$$P_1 + P_2 = (x_3, -y_3),$$

where $x_3 = \lambda^2 - x_1 - x_2$ and $y_3 = \lambda x_3 + \nu$.

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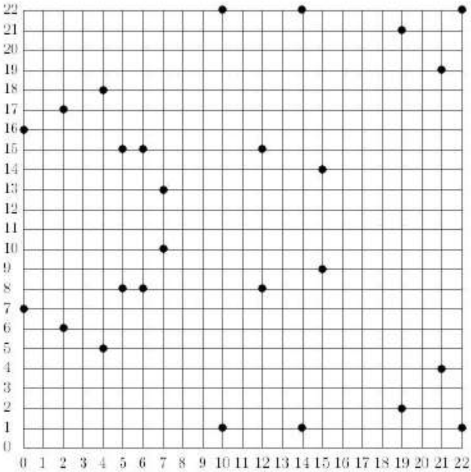
Visualizing Elliptic Curve Cryptography

Elliptic Curves over Finite Fields

Example

Let E be the elliptic curve $y^2 = x^3 + x + 3$ over $\mathbb{F}_{23}$. Then write down all the points of E over $\mathbb{F}_{23}$. Draw the elliptic curve E along with the grid.

Elliptic Curves over Finite Fields



The elliptic curve $y^2 = x^3 + x + 3 \bmod 23$

Elliptic Curves over Finite Fields

Problem

Let E be the elliptic curve $y^2 = x^3 + x + 1$ over $\mathbb{F}_{11}$. Then write down all the points of E over $\mathbb{F}_{11}$. Draw the elliptic curve E along with the grid.

Elliptic Curves over Finite Fields

Solution

NIST's Primes for ECC

$$
\begin{aligned}
p_{192} &= 2^{192} - 2^{64} - 1 \\
p_{224} &= 2^{224} - 2^{96} + 1 \\
p_{256} &= 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1 \\
p_{384} &= 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1 \\
p_{521} &= 2^{521} - 1
\end{aligned}
$$

$$
\begin{aligned}
\text{W-}25519 &= 2^{255} - 19 \\
\text{W-}448 &= 2^{448} - 2^{224} - 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Edwards}25519 &= 2^{255} - 19 \\
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Recommendations for Discrete Logarithm-Based Cryptography: Elliptic Curve Domain Parameters

ElGamal Cryptosystems on Elliptic Curves

- First choose two public elliptic curve points P and Q s/t

$$Q = sP,$$

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- **Decryption:**

$$Dec_k(y_1, y_2) = y_2 - s.y_1$$

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- The plaintext space in general may not consist of the points on the curve E.
- Convert the plaintext as an arbitrary element in $\mathbb{Z}_p$.
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- **Decryption:**
  - Compute $h(kQ)$
  - Compute $c \equiv (y_2 - h(kQ)) \mod p$

ElGamal Cryptosystems on Elliptic Curves

Key Generation

- Let E be an elliptic curve defined over $\mathbb{Z}_p$ (where $p > 3$ is prime) s/t E contains a cyclic subgroup $H = \langle P \rangle$ of prime order n in which the **Discrete Logarithm Problem** is infeasible.
- Let $h : E \to \mathbb{Z}_p$ be a secure hash function.
- Let $\mathcal{P} = \mathbb{Z}_p$ and $C = (\mathbb{Z}_p \times \mathbb{Z}_2) \times \mathbb{Z}_p$. Define

$$\mathcal{K} = \{(E, P, s, Q, n, h) \ : \ Q = sP\},$$

where P and Q are points on E and $s \in \mathbb{Z}_n^*$.

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where P and Q are points on E and $s \in \mathbb{Z}_n^*$.

The values E, P, Q, n, and h are the public key and s is the private key.

ElGamal Cryptosystems on Elliptic Curves

Encryption

- To encrypt a message m sender selects a random number $k \in \mathbb{Z}_n^*$ and compute the ciphertext

$$y = e_K(m, k) = (y_1, y_2) = (\text{ POINT-COMPRESS}(kP), m + h(kQ) \mod p),$$

where $y_1 \in \mathbb{Z}_p \times \mathbb{Z}_2$ and $y_2 \in \mathbb{Z}_p$.

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Decryption

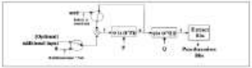
$$d_K(y) = y_2 - h(R) \mod p,$$

where $R = s\text{POINT-DECOMPRESS}(y_1)$.

The Many Flaws of Dual_EC_DRBG

Matthew Green in Dual EC, NSA, RNGs · September 18, 2013 · 3,035 Words

The Many Flaws of Dual_EC_DRBG



The Dual_EC_DRBG generator from NIST SP800-90A.

Update 9/19: *RSA warns developers not to use the default Dual_EC_DRBG generator in BSAFE. Oh lord.*

As a technical follow up to my previous post about the NSA's war on crypto, I wanted to make a few specific points about standards. In particular I wanted to address the allegation that NSA inserted a backdoor into the Dual-EC pseudorandom number generator.

For those not following the story, Dual-EC is a pseudorandom number generator proposed by NIST for international use back in 2006. Just a few months later, Shumow and Ferguson made cryptographic history by pointing out that there might be an NSA backdoor in the algorithm. This possibility — fairly remarkable for an algorithm of this type — looked bad and smelled worse. If true, it spelled almost certain doom for anyone relying on Dual-EC to keep their system safe from spying eyes.

Key Comparison

Symmetric Key Size (in bits)	Based on Factoring (in bits)	Based on DLP (in bits)	Based on ECDLP (in bits)
80	1024	1024	160
112	2048	2048	224
128	3072	3072	256
192	7680	7680	384
256	15360	15360	512

Outline

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- **Fermat factorization:**
  - **The idea:** express n as a difference of two squares:
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Integer Factorization

- **Basic Method:** divide n by all primes $p \leq \sqrt{n}$
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Example

1. Factor $n = 295927$

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$$295927 = 544^2 - 3^2 = 547 \times 541$$

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- Factorize n, with a *high probability*, if any multiple of $\phi(n)$ is known;

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- Write $r = a \cdot 2^s$ with a odd.
- Choose a random b with $1 < b < n-1$.
- If $\gcd(b,n) \neq 1$ we have found a factor of n.

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- Otherwise, let $b_0 \equiv b^a \mod n$. We compute
$$b_1 \equiv b_0^2 \mod n,\ b_2 \equiv b_1^2 \mod n, b_3 \equiv b_2^2 \mod n, \dots$$
- If $b_0 \equiv 1 \mod n$, we choose another b and repeat the procedure.
- Also, if $b_k \equiv -1 \mod n$ for some k, we choose a different b and repeat the procedure.
- If $b_{k+1} \equiv 1 \mod n$ & $b_k \not\equiv \pm 1 \mod n$ for some k,

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So, if the decryption exponent leaks out, changing only e and d is not enough.

Integer Factorization

Example

- Suppose $n = 667, e = 39, d = 79$. We have $(39 \times 79) - 1 = 2^3 \times 385$.
- First select $b = 3$, so $\gcd(3, 667) = 1$.

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$$b_0 = 3^{385} \equiv 162 \mod 667$$

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- We have $b_2 \equiv 1 \mod 667$ & $b_1 \not\equiv \pm 1 \mod 667$.

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b_0 &= 3^{385} \equiv 162 \mod 667 \\
b_1 &= b_0^2 \equiv 231 \mod 667 \\
b_2 &= b_1^2 \equiv 1 \mod 667
\end{aligned}
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- We have $b_2 \equiv 1 \mod 667$ & $b_1 \not\equiv \pm 1 \mod 667$.
- $$\gcd(b_1 - 1, 667) = (230, 667)$$

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- We have $b_2 \equiv 1 \mod 667$ & $b_1 \not\equiv \pm 1 \mod 667$.
- $$\gcd(b_1 - 1, 667) = (230, 667) = 23 \Rightarrow 667 = 23 \times 29$$

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- Choose an integer $a > 1$; let $a = 2$.
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- If $p-1$ has only small prime factors. Then $B!$ is likely to be divisible by $p-1$, say $B! = (p-1)k$. We have

$$b \equiv a^{B!} \equiv \left(a^{p-1}\right)^k \equiv 1 \mod p$$

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$$b \equiv a^{B!} \equiv \left(a^{p-1}\right)^k \equiv 1 \mod p \Rightarrow \gcd(b - 1, n) = p$$

Pollard's $p-1$ method

Algorithm

Input: Integer n to be factored

1. Set $a = 2$ (or some other convenient value)
2. For$\{j = 2, 3, 4, \ldots$ up to a specified bound.$\}\{$
   - (i) Set $a \equiv a^j \mod n$
   - (ii) Compute $d \equiv \gcd(a-1, n)$
   - (iii) If $1 < d < n$ then success, *return* d.

   $\}$
3. Increment j and loop again at Step 2.

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Factor $n = 13927189$ starting with $\gcd(2^{9!} - 1, n)$

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\begin{array}{rcll|rcl}
2^{9!} - 1 & \equiv & 13867883 & \bmod 13927189, & \gcd(2^{9!} - 1, 13927189) & = & 1, \\
2^{10!} - 1 & \equiv & 5129508 & \bmod 13927189, & \gcd(2^{10!} - 1, 13927189) & = & 1, \\
2^{11!} - 1 & \equiv & 4405233 & \bmod 13927189, & \gcd(2^{11!} - 1, 13927189) & = & 1, \\
2^{12!} - 1 & \equiv & 6680550 & \bmod 13927189, & \gcd(2^{12!} - 1, 13927189) & = & 1, \\
2^{13!} - 1 & \equiv & 6161077 & \bmod 13927189, & \gcd(2^{13!} - 1, 13927189) & = & 1, \\
2^{14!} - 1 & \equiv & 879290 & \bmod 13927189, & \gcd(2^{14!} - 1, 13927189) & = & 3823.
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\end{array}$$

$p = 3823$ of n. Thus $q = \frac{n}{p} = \frac{13927189}{3823} = 3643$.

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$$X^2 - Y^2 = (X+Y)(X-Y).$$

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- It is called Fermat factorisation method.

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Factor $n = 25217$ by looking for an integer b making $n + b^2$ a perfect square

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25217 + 1^2 &= 25218 && \textit{not a square},\\
25217 + 2^2 &= 25221 && \textit{not a square},\\
25217 + 3^2 &= 25226 && \textit{not a square},\\
25217 + 4^2 &= 25233 && \textit{not a square},\\
25217 + 5^2 &= 25242 && \textit{not a square},\\
25217 + 6^2 &= 25253 && \textit{not a square},\\
25217 + 7^2 &= 25266 && \textit{not a square},
\end{aligned}$$

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Then we compute

$$25217 = 159^2 - 8^2 = (159 + 8)(159 - 8) = 167 \times 151.$$

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- If n is large, then it is unlikely that a randomly chosen value of b will make $n + b^2$ into a perfect square.

Factorization via Difference of Squares

- If n is large, then it is unlikely that a randomly chosen value of b will make $n + b^2$ into a perfect square.
- It often suffices to write some multiple kn of n as a difference of 2 squares, since if

$$kn = a^2 - b^2 = (a + b)(a - b),$$

then there is a reasonable chance that the factors of n are separated by the right-hand side of the equation.
- n has a nontrivial factor in common with each of $a + b$ and $a - b$.
- Recover the factors by computing $\gcd(n, a + b)$ & $\gcd(n, a - b)$.

Dixon's Factorization Method

- In 1981, John D. Dixon developed this method.
- **The Idea:**
  - Generate a large number of integer pairs (x, y) s/t

$$x^2 \equiv y^2 \mod n,$$

    where $x \neq \pm y \mod n$
  - $x^2 \mod n$ and $y^2 \mod n$ can be completely factorized over the chosen factor base.

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Definition

A positive integer is called B-smooth if none of its prime factors is greater than B.

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Example

- $720 = 2^4 \times 3^2 \times 5^1$; thus 720 is 5-smooth

Dixon's Factorization Method

Example

Factor $n = 84923$ using bound $B = 7$

- Randomly search for integers between $4\lceil \sqrt{n} \rceil = 292$ and n whose squares are B-smooth
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$$513^2 \mod n = 8400 =$$

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  $$537^2 \mod n = 33600 = 2^6 \times 3^1 \times 5^2 \times 7$$
- $(513 \times 537)^2 \mod n = 2^{10} \times 3^2 \times 5^4 \times 7^2 = \left(2^5.3.5^2.7\right)^2 = (16800)^2$
  $\Rightarrow (275481)^2 \equiv (16800)^2 \mod 84923 \Rightarrow (20712)^2 \equiv (16800)^2$

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- $(513 \times 537)^2 \mod n = 2^{10} \times 3^2 \times 5^4 \times 7^2 = \left(2^5.3.5^2.7\right)^2 = (16800)^2$ $\Rightarrow (275481)^2 \equiv (16800)^2 \mod 84923 \Rightarrow (20712)^2 \equiv (16800)^2$
- $84923 = \gcd(20712 - 16800, 84923) \times \gcd(20712 + 16800, 84923)$ $= 163 \times 521$

A Bad Way to Solve DLP

Problem

Find x s/t $y \equiv g^x \mod p$

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- ***Input:*** y
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The worst case $\approx p$ *steps*

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DLP

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2. Create two lists,

   List 1: $e, g, g^2, g^3, \ldots, g^m,$

   List 2: $h, h.g^{-m}, h.g^{-2m}, h.g^{-3m}, \ldots, h.g^{-m^2}.$

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4. Then $x = i + j.m$ is a solution to $g^x = h$.

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Example

Solve the discrete logarithm problem $g^x = h$ in $\mathbb{F}_p^*$ with $g = 9704, h = 13896,\ \&\ p = 17389$.

Shanks's Babystep-Giantstep Algorithm

Example

Solve the discrete logarithm problem $g^x = h$ in $\mathbb{F}_p^*$ with $g = 9704, h = 13896,\ \&\ p = 17389$.

- The number 9704 has order[a] 1242 in $\mathbb{F}_{17389}^*$.
- Set $m = 1 + \lfloor \sqrt{1242} \rfloor = 36$ and

  $u = g^{-m} = 9704^{-36} \equiv 2494 \mod 17389$.

[a] Lagrange's theorem says that the order of g divides $17388 = 2^2.3^3.7.23$. So we can determine the order of g by computing g^n for the 48 distinct divisors of 17388

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Solve the discrete logarithm problem $g^x = h$ in $\mathbb{F}_p^*$ with $g = 9704, h = 13896,\ \&\ p = 17389$.

k	g^k	$h \cdot u^k$	k	g^k	$h \cdot u^k$	k	g^k	$h \cdot u^k$	k	g^k	$h \cdot u^k$
1	9704	347	9	15774	16564	17	10137	10230	25	4970	12260
2	6181	13357	10	12918	11741	18	17264	3957	26	9183	6578
3	5763	12423	11	16360	16367	19	4230	9195	27	10596	7705
4	1128	13153	12	13259	7315	20	9880	13628	28	2427	1425
5	8431	7928	13	4125	2549	21	9963	10126	29	6902	6594
6	16568	1139	14	16911	10221	22	15501	5416	30	11969	12831
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$$13896 \equiv 9704^7.2494^{-32} \equiv 9704^7 \left(9704^{-36}\right)^{-32} \equiv 9704^{1159}$$

Outline

Signature Scheme

Definition

A signature scheme is a five-tuple $(\mathcal{P}, \mathcal{A}, \mathcal{K}, \mathcal{S}, \mathcal{V})$, where the following conditions are satisfied:

Signature Scheme

Definition

A signature scheme is a five-tuple $(\mathcal{P}, \mathcal{A}, \mathcal{K}, \mathcal{S}, \mathcal{V})$, where the following conditions are satisfied:

(i) $\mathcal{P}$ is a finite set of possible messages

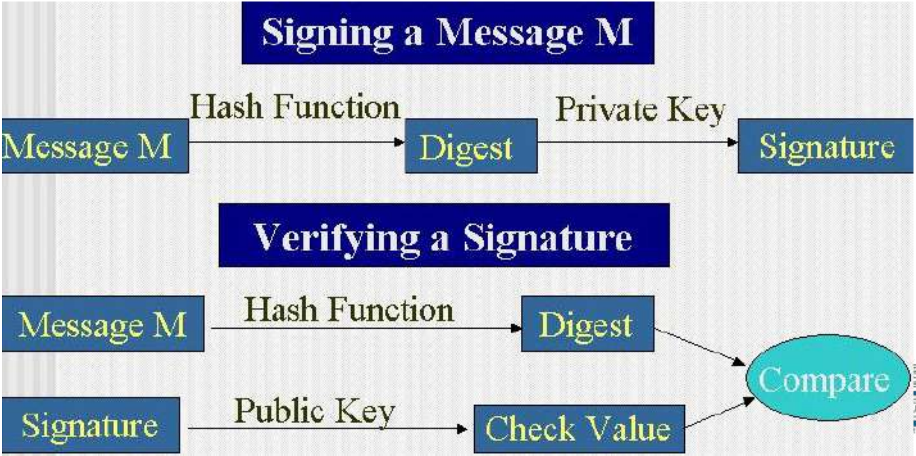
(ii) $\mathcal{A}$ is a finite set of possible signatures

(iii) $\mathcal{K}$, the keyspace, is a finite set of possible keys

(iv) For each $K \in \mathcal{K}$, there is a signing algorithm $sig_K \in \mathcal{S}$ and a corresponding verification algorithm $ver_K \in \mathcal{V}$. Each $sig_K : \mathcal{P} \to \mathcal{A}$ and $ver_K : \mathcal{P} \times \mathcal{A} \to \{true, \ false\}$ are functions s/t the following equation is satisfied for every message $x \in \mathcal{P}$ and for every signature $y \in \mathcal{A}$

$$ver_K = \begin{cases} \text{true} & \text{if} \quad y = sig_K(x) \\ \text{false} & \text{if} \quad y \neq sig_K(x) \end{cases}$$

A pair (x, y) with $x \in \mathcal{P}$ and $y \in \mathcal{A}$ is called a signed message.


Signing a Message M
Message M
Hash Function
Digest
Private Key
Signature
Verifying a Signature
Message M
Hash Function
Digest
Compare
Signature
Public Key
Check Value

RSA Signature Scheme

Signature Generation

A signs a message m. Any entity B can verify A's signature and recover the message m from the signature.

- Compute $\tilde{m} = R(m)$, where $R : \mathcal{M} \to \mathbb{Z}_n$.
- Compute $s \equiv \tilde{m}^d \mod n$.
- A's signature for m is s.

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- A's signature for m is s.

Signature Verification

To verify A's signature s and recover the message m, B should:

- Obtain A's authentic public key (n, e).
- Compute $\tilde{m} \equiv s^e \mod n$.
- Verify that $\tilde{m} \in$ range of $\mathcal{M}$; if not, reject the signature.
- Recover $m = R^{-1}(\tilde{m})$.

DSA

Key Generation

1. Choose a hash function h.
2. Decide a key length L.
3. Choose prime q with with same number of bits as output of h.
4. Choose α-bit prime p such that $q|(p-1)$.
5. Choose g such that $g^q \equiv 1 \mod p$.

Choose x : $0 < x < q$.

Calculate : $y \equiv g^x \mod p$.

(p, q, g, y) $\longrightarrow$ Public Key

x $\longrightarrow$ Private Key

DSA

Signature Generation

1. Generate random k such that $0 < k < q$.
2. Calculate $r \equiv (g^k \mod p) \mod q$.
3. Calculate $s \equiv (k^{-1}(h(m) + xr)) \mod q$.
4. Signature is (r, s).

DSA

Signature Generation

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Signature Verification

1. $w \equiv s^{-1} \mod q$.
2. $u_1 \equiv (h(m).w) \mod q$.
3. $u_2 \equiv rw \mod q$.
4. $v \equiv (g^{u_1}.y^{u_2} \mod p) \mod q$.
5. Verify $v = r$.

Schnorr Signature Scheme

Key Generation

- Let p be a prime s/t the DLP in $\mathbb{Z}_p^*$ is intractable, and let q be a prime and $q \mid (p-1)$. Let $\alpha \in \mathbb{Z}_p^*$ be a q^{th} root of unity modulo p. Let $\mathcal{P} = \{0,1\}^*$, $\mathcal{A} = \mathbb{Z}_q \times \mathbb{Z}_q$, and define

$$\mathcal{K} = \{(p, q, \alpha, a, \beta) \ : \ \beta \equiv \alpha^a \mod p\},$$

  where $0 \le a \le q-1$.

  The values p, q, α, and β are the public key, and a is the private key.

  Finally, let $h \ : \ \{0,1\}^* \to \mathbb{Z}_q$ be a secure hash function.

Schnorr Signature Scheme

Signature Generation

- Signer first selects a (secret) random number $k, \ 1 \leq k \leq q-1$, define

$$sig_K(x,k) = (\gamma, \delta),$$

where

$$\gamma = h(x\|\alpha^k \ mod \ p) \ \& \ \delta = k + a\gamma \ mod \ q.$$

Verification

- For $x \in \{0,1\}^*$ and $\gamma, \delta \in \mathbb{Z}_q$, verification is done by performing the following computations:

$$ver_K(x, (\gamma, \delta)) = true \iff h(x\|\alpha^{\delta}\beta^{-\gamma} \ mod \ p) = \gamma.$$

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The End

Thanks a lot for your attention!