Logic, Proofs, and Counting

Dhananjoy Dey

Indian Institute of Information Technology, Lucknow
ddey@iiitl.ac.in

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Outline

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- It describes a collection of branches of mathematics with the common characteristic that they focus on the study of things consisting of separate, often finite parts.
- It is essential for developing logic and problem-solving abilities.
- A course in discrete mathematics provides the mathematical background needed for all subsequent courses in computer science and for all subsequent courses in the many branches of discrete mathematics.

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- How can I encrypt a message so that no unintended recipient can read it?
- What is the shortest path between two cities using a transportation system?

Goals of This Course

- **Mathematical Reasoning:** Ability to read, understand, and construct mathematical arguments and proofs.
- **Combinatorial Analysis:** Techniques for counting objects of different kinds.
- **Discrete Structures:** Abstract mathematical structures that represent objects and the relationships between them. Examples are sets, permutations, relations, graphs, and trees.

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  An algorithm is a well-defined computational procedure that takes a variable input and halts with an output.

  Algorithmic thinking involves specifying algorithms, analyzing the memory and time required by an execution of the algorithm, and verifying that the algorithm will produce the correct answer.

Discrete Maths in CS, Maths, ...

- **Computer Science:**

Discrete Maths in CS, Maths, . . .

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  The concepts learned will also be helpful in continuous areas of mathematics.
- **Other Disciplines:** It is also useful in courses in philosophy, economics, linguistics, and other disciplines.

Outline

Syllabus

- Logic, Proofs, and Counting
- Basic Structures
- Introduction to Abstract Algebra
- Introduction to Number Theory
- Introduction to Graph Theory

References

- **Textbook**

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Discrete Mathematics and Its Applications, McGraw-Hill Education, Eighth Edition, 2019.

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- Owen D. Byer, Deirdre L. Smeltzer, & Kenneth L. Wantz, *Journey into Discrete Mathematics*, AMS/MAA Textbooks, volume 41, 2018.
- Harry Lewis, & Rachel Zax, *Essential Discrete Mathematics for Computer Science*, Princeton University Press, 2019.
- Gerard O'Regan, *Guide to Discrete Mathematics: An Accessible Introduction to the History, Theory, Logic and Applications*, Springer 2016.
- Kenneth H. Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Second Edition, 2018.

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4. $x + 3 = 7$.

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Example (Not Propositions)

1. What is the time now?
2. $x + y = z$

Propositional Logic

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- These rules are used to distinguish between valid and invalid mathematical arguments.
- A major goal of Discrete Maths is to learn how to understand and how to construct correct mathematical arguments
- We begin our study of discrete mathematics with an introduction to logic.
- In mathematics, ‘logic’ is used to refer to a particular type of formal reasoning.

Propositional Logic

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  - **Propositional Variables:** $p, q, r, s, \ldots$

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Propositional Logic

- **Constructing Propositions**
  - **Propositional Variables:** $p, q, r, s, \ldots$
  - The proposition that is always *true* is denoted by T and the proposition that is always *false* is denoted by F.
  - **Compound Propositions** – constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$ or $\implies$
    - Biconditional $\leftrightarrow$ or $\iff$

Compound Propositions: Negation

- Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.
  - The **negation** of a proposition p is denoted by $\neg p$

p	**¬p**
T	F
F	T

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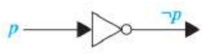

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p – you are students of 2^{nd} year B.Tech

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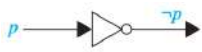

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p – you are students of 2^{nd} year B.Tech

$\neg p$ – you are not students of 2^{nd} year B.Tech

Remark: Other notations for negation are $\bar{p}, \sim p, -p, Np, p'$ or $!p$.

Conjunction

- The conjunction of propositions p and q is denoted by $p \wedge q$

p	**q**	**p ∧ q**
T	T	T
T	F	F
F	T	F
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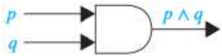

Example

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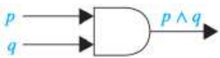

Example

p – you are attending this lecture
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$p \wedge q$ – you are attending this lecture *and* it is sunny today

Disjunction

- The conjunction of propositions p and q is denoted by $p \vee q$

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{p \vee q}$
T	T	T
T	F	T
F	T	T
F	F	F

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Disjunction

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p	q	p ∨ q
T	T	T
T	F	T
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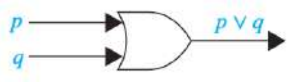

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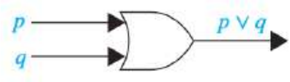

Example

p – you are attending this lecture
q – you are watching your mobile
$p \vee q$ – you are attending this lecture *or* watching your mobile

Inclusive or/Exclusive or (Xor)

- In English ‘or’ has two distinct meanings.
  - **Inclusive or –** “Students who have taken Theory of Computation or Cryptography class may take this class,"

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    This is the meaning of **disjunction**.
  - **Exclusive or (Xor) –** “Soup or salad comes with the main course of your lunch," you do not expect to be able to get both soup and salad.

    This is the meaning of **Exclusive Or (Xor)**.

    It is denoted by $\oplus$. E.g., $p \oplus q$, one of p and q must be true, but not both.

Exclusive or (Xor)

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
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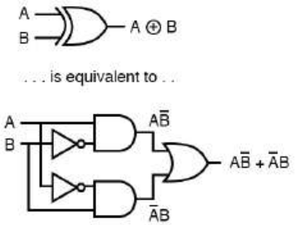

Exclusive or (Xor)

A	B	A ⊕ B
T	T	F
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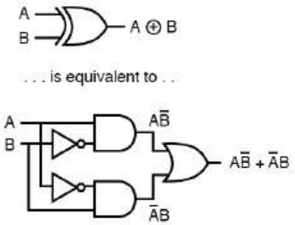

Theorem

$$p \oplus q \iff (p \wedge \neg q) \vee (\neg p \wedge q).$$

Conditional Statements: Implication

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  - If color the moon is green, then you have more money than Gautam Adani.
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- One way to view the logical conditional is to think of an **obligation** or **contract**.
  - If you get 85% on the final, then you will get an A grade.

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1. If the student is good in mathematics, then he is humble.

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1. If the student is good in mathematics, then he is humble.
2. The student is humble, if he is good in mathematics.
3. The student is good in mathematics implies that he is humble.
4. The student is good in mathematics only if he is humble.
5. To be humble is necessary for the student to be good in mathematics.
6. The student's being good in mathematics is sufficient to conclude that he is humble.

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(i) If S then T.

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For two statements S and T the following convey the same meaning:

(i) If S then T.
(ii) T if S.
(iii) S implies T.
(iv) S only if T.
(v) T is necessary for S.
(vi) S is sufficient for T.

Converse, Contrapositive, and Inverse

- From $p \Rightarrow q$ we can form new conditional statements
  - $q \Rightarrow p$ is the **converse** of $p \Rightarrow q$
  - $\neg q \Rightarrow \neg p$ is the **contrapositive** of $p \Rightarrow q$
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- We first show that the contrapositive, $\neg q \Rightarrow \neg p$, of a conditional statement $p \Rightarrow q$ always has the same truth value as $p \Rightarrow q$.
- Note that the contrapositive is false only when
  - $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false.
- Neither the converse, $p \Rightarrow q$, nor the inverse, $\neg p \Rightarrow \neg q$, has the same truth value as $p \Rightarrow q$ for all possible truth values of p and q.

Converse

- Consider the two implications
  (i) If the student is sincere, then he is humble.
  (ii) If the student is humble, then he is sincere.

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  The student is humble if and only if he is sincere.

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Example

For real numbers x and $a > 0$, consider the statements "$|x| < a$" and "$x \in (-a, a)$".
Then the two statements "if $|x| < a$, then $x \in (-a, a)$" and "if $x \in (-a, a)$, then $|x| < a$" are converses of each other.

Note that the two statements can also be written as

$$|x| < a \Leftrightarrow x \in (-a, a)$$

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- Hence, a *conditional statement* and *its contrapositive* are **equivalent**.

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- Hence, a *conditional statement* and *its contrapositive* are **equivalent**.
- The *converse* and the *inverse* of a conditional statement are also equivalent.
  However neither is equivalent to the original conditional statement.

Theorem

$$p \Rightarrow q \Leftrightarrow \neg p \vee q.$$

Biconditional/Equivalence

- If p and q are propositions, then we can form the biconditional proposition $p \Leftrightarrow q$, read as
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T	T	T
T	F	F
F	T	F
F	F	T

Table: Truth Table

- Some alternative ways "p iff q" is expressed in English:
  - p is necessary and sufficient for q
  - if p then q, and conversely

Propositional Logic

Example	Name	Meaning
$\neg p$	Negation	Not p
$p \vee q$	(Inclusive) Or	Either p or q or both
$p \wedge q$	And	Both p and q
$p \oplus q$	XOR	Either p or q, but not both
$p \Rightarrow q$	Implies	If p, then q
$p \Leftrightarrow q$ / $p \iff q$	Biconditional / Equivalence	p if and only if q

Truth Tables for Compound Propositions

- A truth table presents the truth values of a compound propositional formula in terms of the truth values of the components.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\Rightarrow$	4
$\Leftrightarrow$	5

Example of Truth Table

Construct a truth table for $p \vee q \Rightarrow \neg r$

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p	q	r	$\neg r$	$p \vee q$	$p \vee q \Rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Tautologies, Contradictions, and Contingencies

Definition

- *A **tautology** is a proposition which is always true.*

$$p \vee \neg p$$

- *A **contradiction** is a proposition which is always false.*

$$p \wedge \neg p$$

- *A **contingency** is a proposition which is neither a tautology nor a contradiction.*

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Truth table for De Morgan's Second Law:

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
T	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$
- Idempotent laws: $p \wedge p \equiv p$, $p \vee p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$
- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

Key Logical Equivalences

- Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
  $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws: $p \vee (p \wedge q) \equiv p$
  $p \wedge (p \vee q) \equiv p$

Logic Puzzles

- In Lucknow, there are two kinds of inhabitants, Type-1, who always tell the truth, and Type-2, who always lie.
- You come to Lucknow and meet A and B.
  - A says "B is a Type-1."
  - B says "The two of us are of opposite types."

Exercise

What are the types of A and B?

Logic Puzzles

Outline

1. Introduction
   - Syllabus
     - References

2. The Foundations: Logic and Proofs
   - Propositional Logic
   - Proofs
     - Direct Proof
     - Proof by Contradiction
     - Proof by Contrapositive
     - Constructive Proofs, Counterexamples, and Vacuous Proofs

3. Counting

Proofs of Mathematical Statements

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  - Easier for to understand and to explain to people.
  - However, it is also easier to introduce errors.
- Proofs have many practical applications:
  - verification that computer programs are correct
  - establishing that operating systems are secure
  - enabling programs to make inferences in artificial intelligence
  - showing that system specifications are consistent

Some Terminology

- A **theorem**

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- Less important theorems are sometimes called **propositions**.

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Conversion of Plain English into Mathematical Form

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Everybody loves somebody

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For every person A, *there is a* person B such that (or $\ni$) A loves B.

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- The phrases '*for all*', '*for any*', '*for every*', '*for some*', & '*there exists*' are called **quantifiers**
- Their careful use is an important part in mathematics.
- The symbol $\forall$ stands for '*for all*', '*for any*', or '*for every*'
- The symbol $\exists$ stands for '*there exists*' or '*for some*'.

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1. (i) S_1 : In every shelf in the library there is a mathematics book.
2. (ii) S_2 : There is a shelf in the library in which all books are story books.

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Given any shelf in the library, it has a non-story book

Forms of Theorems

- Many theorems assert that a property holds for all elements in a domain
- Often the *universal quantifier* (needed for a precise statement of a theorem) is omitted by standard mathematical convention.

Example

The statement:

If $x > y > 1$, where x & y are positive real numbers, then $x^2 > y^2$

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The statement:

If $x > y > 1$, where x & y are positive real numbers, then $x^2 > y^2$

really means

For all positive real numbers x & y, if $x > y > 1$, then $x^2 > y^2$.

Proving Theorems

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- Many theorems have the form:

$$\forall\ x\ (P(x) \Rightarrow Q(x))$$

- To prove them, we show that where c is an arbitrary element of the domain,

$$P(c) \Rightarrow Q(c)$$

- By universal generalization the truth of the original formula follows.
- So, we must prove something of the form: $p \Rightarrow q$.

Theorem

Every odd integer is equal to the difference between the squares of two integers.

Methods of Proof

- **Direct Proof**
- **Proof by Contradiction**
- **Proof by Contrapositive**
- **Constructive Proofs, Counterexamples, and Vacuous Proofs**
- **Mathematical Induction**

Direct Proof

- To prove a statement of the form “if A, then B" directly, begin by assuming that A is true.
- Then, making use of *axioms*, *definitions*, *previously proven theorems*, and *rules of inference*, proceed directly until B is reached as a conclusion.
- Direct proofs are most easily employed when establishing the general form of the antecedent is straightforward.

Example

Theorem

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For any integer n, n^2 is odd iff n is odd.

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The statement "n^2 is odd iff n is odd" is really two statements in one:

1. if n is odd then n^2 is odd
2. if n^2 is odd then n is odd

Example

Proof by Contradiction

- The technique known as proof by contradiction is one type of **indirect proof**.
- In a proof by contradiction, in order to prove a statement of the form "If A, then B", one assumes that both A and $\neg$B are true.

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- That is, whenever A is true, B must also be true.
- This method of proof is useful when assuming ¬B allows you to easily utilize a definition or theorem.

Example

Only if part of previous theorem:

Proof.

Now, we have to show that if n^2 is odd, then n must be odd.

Suppose this is not true for all n, and that n is a particular integer s/t n^2 is odd but n is not odd.

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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \\ &= 2.j, \qquad \textit{where } j = 2k^2 \end{aligned}$$

Thus, n^2 is even which contradicts our assumption.

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Thus, n^2 is even which contradicts our assumption.

That is, the assumption, n is an integer s/t n^2 is odd but n is not odd, was false.
So its negation is true: *if n^2 is odd, then n is odd*. □

Corollary

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If n is odd, then n^4 is odd.

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Proof.

Note that $n^4 = (n^2)^2$.

Since n is odd, by previous theorem, n^2 is also odd.

Then since n^2 is odd, again the theorem, n^4 is odd. □

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- Referencing *axioms*, *definitions*, *previously proven theorems*, and *rules of inference*, the proof ultimately reaches the conclusion that $\neg A$ is true.
- In other words, this is a direct proof on the contrapositive of the original statement $A \Rightarrow B$.

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Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

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3. We find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.
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3. We find that $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$.
4. This tells us that $3n+2$ is even.
5. This is the negation of the premise of the theorem.

We have proved that if $3n+2$ is odd, then n is odd. □

Constructive Proofs

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- While proofs of universally quantified statements are more commonly encountered, knowing how to prove an existentially quantified statement is essential.
- Recall that an existentially quantified statement simply makes a claim about the existence of a particular entity.
- If a single example of the desired object can be produced, the statement has been proven.
- Such a proof is often called a constructive proof.

Example

Exercise

Prove that there exists an integer n *s/t*

$$\frac{n^2 + n}{3n + 8} = 1.$$

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Solution

- *First Thoughts – find such* n
- *Prove the statement for those* n.

Counterexamples

- One is presented with a statement that may or may not be true and is asked to prove or disprove the given statement.
- In this case, experimentation may be required in order to decide whether to attempt a proof or a disproof.

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- In this case, experimentation may be required in order to decide whether to attempt a proof or a disproof.
- To disprove a universally quantified statement, providing a single **counterexample** is sufficient.
- Thus disproof of a universally quantified statement is constructive.
- On the other hand, disproving an existentially quantified statement amounts to proving a quantified statement:

  one must show that the given statement does not hold for any elements of the domain of discourse.

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Prove that the irrational numbers are not closed under multiplication.

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First Thoughts. *The statement p : irrational numbers are closed under multiplication is a universal statement.*

$\neg p$: *It is not the case that the irrational numbers are closed under multiplication.*

This means the given statement is logically equivalent to an existential statement.

We can prove it false if we can produce two irrational numbers whose product is rational.

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This means the given statement is logically equivalent to an existential statement.

We can prove it false if we can produce two irrational numbers whose product is rational.

Let $x = \sqrt{2}$ & $y = \sqrt{8}$. Then x & y are both irrational, but $xy = 4$ is rational.

Thus the irrational numbers are not closed under multiplication.

Counterexamples

In summary,

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- A single example cannot prove a universally quantified statement (unless the domain of discourse contains only one element);
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- a single example can prove an existentially quantified statement;
- a single counterexample cannot disprove an existentially quantified statement (unless the domain of discourse contains only one element).

Vacuous Proofs

- Now, we consider the situation in which a statement of the form "if A, then B" is to be proven, but the statement A is never true.
- Since a conditional statement is always true when the antecedent is false.
- We would regard such a statement as vacuously true.

Example

Exercise

For all $x \in \mathbb{R}$, if $x^2 < 0$ then $3x^2 + 5 = -7x$

Solution

For any $x \in \mathbb{R}, x^2 \geq 0$.

Thus, since the antecedent ($x^2 < 0$) is always false, the implication is ***vacuously true****.*

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3. *Prove it for* $n = k + 1$

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$$[P(1) \wedge \forall k\ (P(k) \Rightarrow P(k+1))] \Rightarrow \forall n\ P(n).$$

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- Let $P(n)$ be the statement that n can be written as the product of prime numbers.
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- For $n = 2$, $P(n)$ is true.
- Now, assume that for some $k \geq 2$, each integer n with $2 \leq n \leq k$ may be written as a product of primes. We need to prove that $k + 1$ is a product of primes.

□

Proof by Mathematical Induction

Proof.

- **Case (a):** Suppose $k + 1$ is a prime. Then we are done.

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$$k+1 = a \cdot b.$$

By the strong inductive hypothesis, since $2 \leq a, b \leq k$, both a & b are the product of primes. Thus,

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This is proved by strong induction. □

Mathematical Induction

Induction

Math Induction

Weak | **Strong**

Mathematical Induction

Induction

Math Induction

Weak **Strong**

Definition

- ***Weak Induction:*** $[P(1) \land \ \forall\, k\ (P(k) \Rightarrow P(k+1))] \Rightarrow \ \forall\, n\ P(n).$

Mathematical Induction

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Weak **Strong**

Definition

- ***Weak Induction:*** $[P(1) \wedge \ \forall\, k\, (P(k) \Rightarrow P(k+1))] \Rightarrow \ \forall\, n\, P(n).$
- ***Strong Induction:*** $[P(1) \wedge \ \forall\, k (P(1) \wedge P(2) \wedge \ldots \wedge (P(k) \Rightarrow P(k+1))] \Rightarrow \ \forall\, n\, P(n).$

Importance of Base Step

Example

- Consider a statement $P(n)$ as $2 + 4 + \ldots + 2n = (n+2)(n-1)$.
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Note:

- Observe that $P(1)$ is true
- Let $k \geq 1$ and assume that $P(k)$ is true. Show that $P(k + 1)$ is true.

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$$\begin{aligned}(k+1)+1 &< k+1 \\ k+2 &< k+1\end{aligned}$$

- Thus, induction step is true.

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- Now, we try to prove this is true for $n = k + 1$.

$$(k + 1) + 1 \quad < \quad k + 1$$

$$k + 2 \quad < \quad k + 1$$

- Thus, induction step is true.
- However, it is not true for $n = 1$.

Thus, the given inequality is not true.

Arbitrary Base Step

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Definition

Let $A \subset \mathbb{Z}$ *and* $N \in \mathbb{Z}$. *Assume that*

(i) $N \in A$

(ii) *for* $k \geq N$, $k \in A$ *implies* $k + 1 \in A$.

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Exercise

Prove that $n! > 2n$ *for all positive integers* $n \geq 4$. *(The base case here is 4.)*

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks.

- There are n_1 ways to do the first task and n_2 ways to do the second task.
- Then there are $n_1 \times n_2$ ways to do the procedure.

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How many different number plates can be made if each plate contains a sequence of 2 uppercase English letters followed by 4 digits?

Solution

There are $26^2 \times 10^4$ *many different number plates*

Counting Functions

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How many one-to-one functions are there from a set with m elements to a set with n elements?

Solution

There are $n(n-1)(n-2)\ldots(n-m+1)$ *such functions.*

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The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

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Example

The IIITL must choose either a student from CS, a student from CSAI, a student from CSAI, or a student from IT as a representative for students' committee.

The Sum Rule

Counting Passwords

Exercise

A password consists of 6 to 8 characters, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible ways you can choose your passwords?

Counting Passwords

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways,

then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- This is also known as, the principle of *inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Exercise

How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

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Solution

- *Number of bit strings of length 8 that start with a 1 bit:* $2^7 = 128$
- *Number of bit strings of length 8 that end with bits 00:* $2^6 = 64$
- *Number of bit strings of length 8 that start with a 1 bit and end with bits 00 :* $2^5 = 32$

Thus, the number is $128 + 64 - 32 = 160$.

The Pigeonhole Principle

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If you want to place n *pigeons into* m *pigeonholes, and* $n > m$*, then at least one pigeonhole will contain more than one pigeon.*

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Proof.

- Suppose none of the m pigeonholes, has more than one pigeon.
- Then the total number of pigeons would be at most m.
- This contradicts the statement that we have n pigeons and $n > m$.

Thus, our assumption was wrong. Hence proved!

□

The Pigeonhole Principle

Corollary

A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

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Among any group of 366 people, there must be at least 2 having the same birthday.

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Problem

Let there be $m+1$ people $\{P_1, P_2, \ldots, P_{m+1}\}$ in a room. What should be the value of m so that the probability that atleast one of the persons $\{P_2, P_3, \ldots, P_{m+1}\}$ shares birthday with P_1 is greater than $\frac{1}{2}$?

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Problem

How many people must be there in a room, so that the probability of atleast 2 of them sharing the same birthday is greater than $\frac{1}{2}$?

The Pigeonhole Principle

Theorem

Let A be a finite set, partitioned into finite subsets $S_1, S_2, \ldots, S_m$. If $|A| = n > m$, then at least one of these m subsets contains more than one element.

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Principle (Generalized Pigeonhole)

If you want to place n pigeons into m pigeonholes with respective capacities of $c_1, c_2, \ldots, c_m$ and $n > c_1 + c_2 + \ldots + c_m$ then at least one of the pigeonholes will contain more pigeons than its capacity.

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Principle (Extended Pigeonhole)

If you want to place n pigeons into m pigeonholes, then one of the pigeonholes will contain at least $\lfloor \frac{n-1}{m} \rfloor + 1$ pigeons.

The Pigeonhole Principle

Exercise

1. *Prove that in any set of 99 natural numbers, there is a subset of 15 of them with the property that the difference of any two numbers in the subset is divisible by 7.*

The Pigeonhole Principle

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1. *Prove that in any set of 99 natural numbers, there is a subset of 15 of them with the property that the difference of any two numbers in the subset is divisible by 7.*
2. *There are 75 students in a class. Each got an A, B, C, or D on a test. Show that there are at least 19 students who received the same grade.*

The End

Thanks a lot for your attention!