Public Key Cryptography

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1

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Outline

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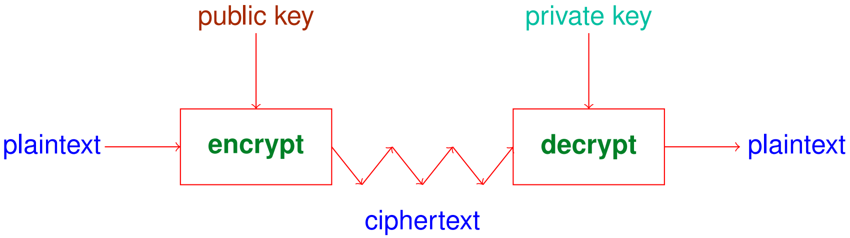
A Generic View of Public Key Crypto



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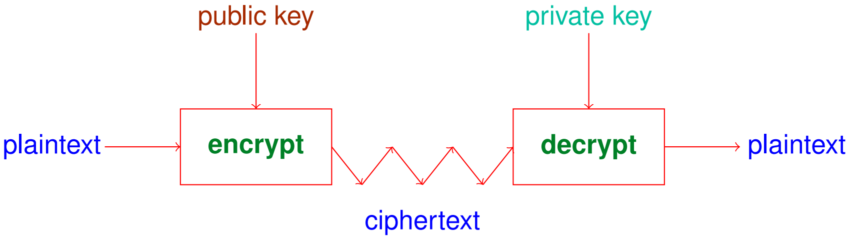

Advantages over symmetric-key

1. Better key distribution and management
   - No danger that public key compromised
2. New protocols
   - Digital Signature
3. Long-term encryption

Only disadvantage:

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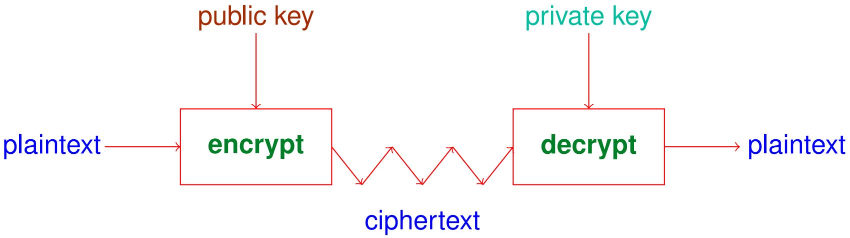

Advantages over symmetric-key

1. Better key distribution and management
   - No danger that public key compromised
2. New protocols
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3. Long-term encryption

Only disadvantage: much more slower than symmetric key crypto

Definition

PKC

A public key cryptosystem is a pair of families $\{E_k : k \in \mathcal{K}\}$ and $\{D_k : k \in \mathcal{K}\}$ of algorithms representing invertible transformations,

$$E_k : \mathcal{M} \to \mathcal{C} \;\&\; D_k : \mathcal{C} \to \mathcal{M}$$

on a finite message space $\mathcal{M}$ and ciphertext space $\mathcal{C}$, such that

(i) for every $k \in \mathcal{K},\ D_k$ is the inverse of E_k and vice versa,

(ii) for every $k \in \mathcal{K},\ M \in \mathcal{M}$ and $C \in \mathcal{C}$, the algorithms E_k and D_k are *easy* to compute.

(iii) for almost every $k \in \mathcal{K}$, each easily computed algorithm equivalent to D_k is computationally infeasible to derive from E_k,

(iv) for every $k \in \mathcal{K}$, it is feasible to compute inverse pairs E_k and D_k from k.

Definition

Computationally Infeasible

A task is computationally infeasible if either the time taken or the memory required for carrying out the task is finite but impossibly large.

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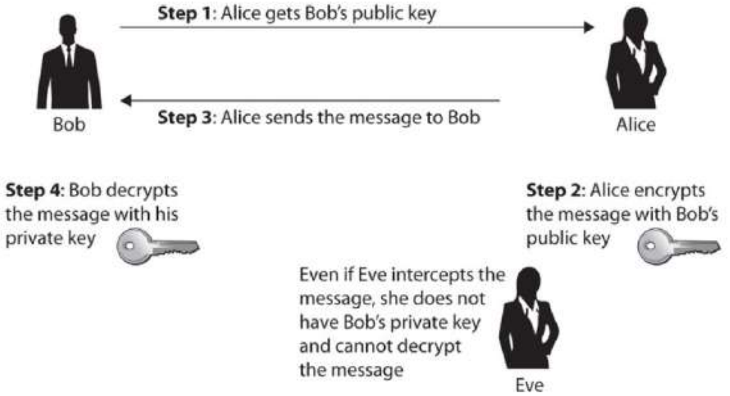
Any computational task which takes $\geq 2^{112}$ bit operations, we say, it is computationally infeasible in present day scenario.

PKC



Step 1: Alice gets Bob's public key
Step 3: Alice sends the message to Bob
Bob
Alice
Step 4: Bob decrypts the message with his private key
Step 2: Alice encrypts the message with Bob's public key
Even if Eve intercepts the message, she does not have Bob's private key and cannot decrypt the message
Eve

Digital Signature

Signing a Message M

Message M

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Message M $\xrightarrow{Hash\ Function\ h}$ Digest $h(M)$

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Message M $\xrightarrow{\textit{Hash Function } h}$ Digest $h(M)$ $\xrightarrow{\textit{Private Key}}$ Signature

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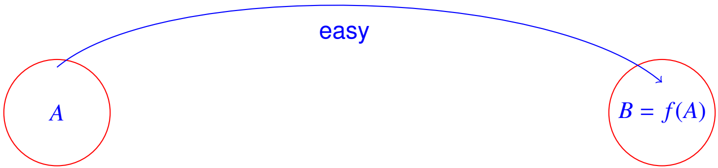
One-way Function



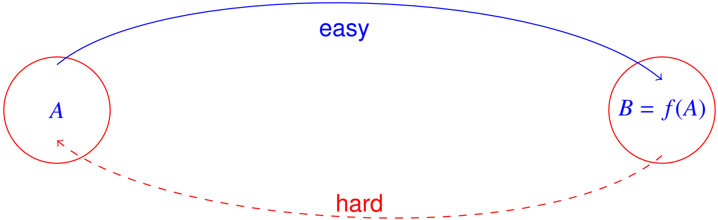
Definition

Easy: $\exists$ *a polynomial-time algorithm that, on input* $m \in A$ *outputs* $c = f(m)$.

Definition

Hard: *Every probabilistic polynomial-time algorithm trying, on input* $c(= f(m))$ *to find an inverse of* $c \in B$ *under* f, *may succeed only with negligible probability.*

One-way Function



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Examples of One-way Function

- Cryptographic hash functions, viz., SHA-2 and SHA-3 (Keccak) family.
- The function

$$f : \mathbb{Z}_p \to \mathbb{Z}_p,$$

$$x \mapsto x^{2^{24}+17} + a_1.x^{2^{24}+3} + a_2.x^3 + a_3.x^2 + a_4.x + a_5,$$

where $p = 2^{64} - 59$ and each a_i $(\in \mathbb{Z}_p)$ is 19-digit number for $1 \leq i \leq 5$.

Trapdoor One-way Function

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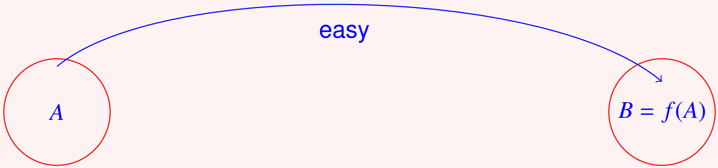

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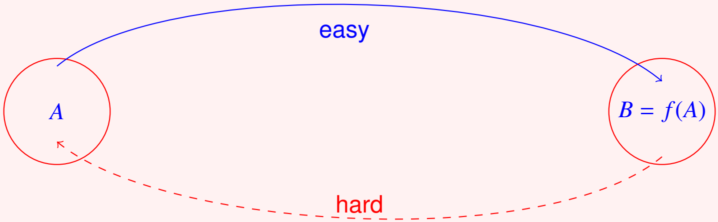

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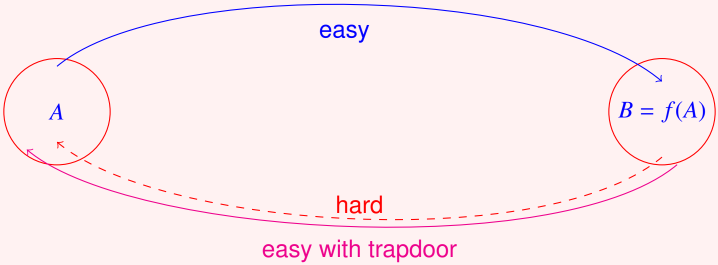

Trapdoor One-way Function

Definition

A trapdoor one-way function is a one-way function $f : \mathcal{M} \rightarrow \mathcal{C}$, satisfying the additional property that $\exists$ some additional information or trapdoor that makes it easy for a given $c \in f(\mathcal{M})$ to find out $m \in \mathcal{M} : f(m) = c$, but without the trapdoor this task becomes hard.

Examples Trapdoor One-way Function

- **Integer Factorization:** Given $n \in \mathbb{Z}^+$, find $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ where the p_i are pairwise distinct primes and each $e_i \geq 0$ for $1 \leq i \leq k$. $\rightarrow$ **hard problem**.

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- Consider the number 37015031

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- Consider the number 37015031$= 6079 \times 6089$
- Consider the number 96679789$= 9743 \times 9923$

Examples Trapdoor One-way Function

- **Discrete Logarithm Problem:** Given an abelian group $(G, \ .)$ and $g \in G$ of order n. Given $h \in G$ such that $h = g^x$ find x ($DLP(g, h) \rightarrow x$). $\rightarrow$ **hard problem**.

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Example

- Let $p = 97$. Then $\mathbb{Z}_{97}^*$ is a cyclic group of order $n = 96$.
  5 is a generator of $\mathbb{Z}_{97}^*$.
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- **Computational Diffie-Hellman Problem:** Given $a = g^x$ and $b = g^y$ find $c = g^{xy}$. ($CDH(g, a, b) \rightarrow c$). $\rightarrow$ **hard problem**.

Example Trapdoor One-way Function

- **Computational Diffie-Hellman Problem:** Given $a = g^x$ and $b = g^y$ find $c = g^{xy}$. ($CDH(g, a, b) \rightarrow c$). $\rightarrow$ **hard problem**.
- **Elliptic Curve Discrete Logarithm Problem (ECDLP):** $\mathbb{E}$ denotes the collections of points on a elliptic curve and $P \in \mathbb{E}$. Let $\mathcal{S}$ be the cyclic subgroup of $\mathbb{E}$ generated by P. Given $Q \in \mathcal{S}$, find an integer x such that $Q = x.P$. $\rightarrow$ **hard problem**.

Outline

DH Key Exchange

DH Key Exchange

Both parties know p and g

Alice

1. Alice generates a
2. Alice's public value is $g^a \bmod p$
3. Alice computes $g^{ab} = (g^b)^a \bmod p$

Bob

1. Bob generates b
2. Bob's public value is $g^b \bmod p$
3. Bob computes $g^{ba} = (g^a)^b \bmod p$

Since $g^{ab} = g^{ba}$ they now have a shared secret key usually called k $(K = g^{ab} = g^{ba})$

DH Key Exchange

- k is the shared secret key.
- Knowing $g,\ g^a\ \&\ g^b$, it is hard to find g^{ab}.
- **Idea of this protocol:** The enciphering key can be made public since it is computationally infeasible to obtain the deciphering key from enciphering key.
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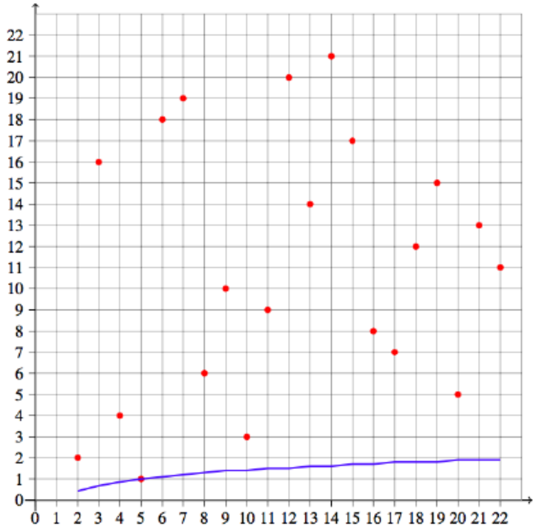
- PKCS #3 (Version 1.4): Diffie-Hellman Key-Agreement Standard, An RSA Laboratories Technical Note – Revised November 1, 1993.

Discrete Logarithm $\bmod 23$ to the Base 5



- Clifford Cocks, Malcolm Williamson & James Ellis developed Non-secret Encryption between 1969 and 1974.

Clifford Cocks, Malcolm Williamson, and James Ellis.

- All were at GCHQ, so this stayed secret until 1997.

Chinese Remainder Theorem

Theorem

Suppose $m_1, m_2, \cdots, m_r \in \mathbb{Z}^+ \; : \; gcd(m_i, m_j) = 1$ *for* $i \neq j$.
Then $x \equiv a_i \mod m_i$ *has* ! *solution* $\mod M (= \prod_{i=1}^{r} m_i)$, *which is given by*

$$x \equiv \sum_{i=1}^{r} a_i.M_i.y_i \mod M,$$

where $M_i = \frac{M}{m_i}$ & $y_i = M_i^{-1} \mod m_i$ *for* $1 \leq i \leq r$.

Chinese Remainder Theorem

Problem

Find x *s/t*

$x \equiv 5 \mod 7$, $x \equiv 3 \mod 11$, $x \equiv 10 \mod 13$

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Non-secret Encryption

Key Generation

1. Select 2 large distinct primes p & q such that $p \nmid q-1$ and $q \nmid p-1$.

   Public key: $n = pq$.

2. Find numbers r & s, s/t $p.r \equiv 1 \mod (q-1)$ and $q.s \equiv 1 \mod (p-1)$.
3. Find u & v, s/t $u.p \equiv 1 \mod q$ and $v.q \equiv 1 \mod p$.

   Private key: (p, q, r, s, u, v).

Non-secret Encryption

Encryption

$$C \equiv M^n \mod n \text{ for } 0 \leq M < n.$$

Decryption

1. $a \equiv C^s \mod p$ and $b \equiv C^r \mod q$.
2. $M \equiv a.q.v + b.p.u \mod n$.

Modular Exponentiation by The Repeated Squaring I

Compute $b^n \mod m$

1. Use a to denote the partial product.
2. We'll have $a \equiv b^n \mod m$.
3. We start out with $a = 1$.
4. Let $n_0, n_1, \ldots n_{k-1}$ denote the binary digits of n, i.e.,

$$n = n_0 + 2n_1 + 4n_2 + \ldots + 2^{k-1} n_{k-1}.$$

5. If $n_0 = 1$, change a to b (otherwise keep $a = 1$).
   Then set $b_1 = b^2 \mod m$
6. If $n_1 = 1$, multiply a by b_1 (and reduce $\mod m$); otherwise keep a unchanged.
7. Next square b_1, and set $b_2 = b_1^2 \mod m$

Modular Exponentiation by The Repeated Squaring II

8. If $n_2 = 1$, multiply a by b_2 (and reduce $\mod m$); otherwise keep a unchanged.
9. Continue in this way. You see that in the j-th step you have computed $b_j \equiv b^{2^j} \mod m$.
10. If $n_j = 1$, i.e., if 2^j occurs in the binary expansion of n, then you include b_j in the product for a (if 2^j is absent from n, then you do not).
11. It is easy to see that after the $(k-1)$-st step you'll have the desired

$$a \equiv b^n \mod m.$$

$$\text{Time}(b^n \mod m) = O((\log n)(\log^2 m)).$$

Modular Exponentiation by The Repeated Squaring

Example

Let us compute $5^{100} \mod 33$.

Modular Exponentiation by The Repeated Squaring

Example

Let us compute $5^{100} \mod 33$.

$$
\begin{aligned}
5^1 &= 5 \\
5^2 &= 25 \\
5^4 &= 25 \times 25 \equiv 31 \mod 33 \\
5^8 &\equiv 31 \times 31 \equiv 4 \mod 33 \\
5^{16} &\equiv 4 \times 4 \equiv 16 \mod 33 \\
5^{32} &\equiv 16 \times 16 \equiv 25 \mod 33 \\
5^{64} &\equiv 25 \times 25 \equiv 31 \mod 33 \\
5^{96} &\equiv 31 \times 25 \equiv 16 \mod 33 \\
5^{100} &\equiv 16 \times 31 \equiv 1 \mod 33
\end{aligned}
$$

Outline

RSA Key Generation

- Generate two large distinct random primes p & q.
- Compute $n = pq$ and $\phi(n) = (p-1)(q-1)$.
- Select a random integer $e,\ 1 < e < \phi(n)$ s/t $\gcd(e, \phi(n)) = 1$.
- Compute the unique integer $d,\ 1 < d < \phi(n)$ s/t

$$ed \equiv 1 \mod \phi(n).$$

Public key is (n, e); Private key is (p, q, d).

RSA Encryption/Decryption

Encryption:

$$c \equiv m^e \mod n,$$

Plaintext m and ciphertext $c \in \mathbb{Z}_n$.

Decryption:

$$m' \equiv c^d \mod n.$$

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PKCS #1 v2.2: RSA Cryptography Standard, RSA Laboratories – October 27, 2012.

RSA Validation

We have

$$c^d \equiv (m^e)^d \equiv m^{ed} \equiv m^{1+k.\phi(n)} \mod n,$$

since $ed \equiv 1 \mod \phi(n)$, where k is an integer.

RSA Validation

RSA Validation

Strong Prime Number

Definition

A prime p is called a strong prime if

(i) $p - 1$ *has a large prime factor, say* r,

(ii) $p + 1$ *has a large prime factor, and*

(iii) $r - 1$ *has a large prime factor.*

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Properties of Euler phi function

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$$\phi(mn) = \phi(m)\phi(n).$$

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ii. The Euler phi function is multiplicative. That is, if $gcd(m, n) = 1$, then

$$\phi(mn) = \phi(m)\phi(n).$$

iii. If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, is the prime factorization of n, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

Modular Arithmetic

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- Let n be an odd composite integer. An integer $a, \ 1 \leq a \leq n-1, \ni \ a^{n-1} \not\equiv 1 \mod n$ is called a **Fermat witness** (to compositeness) for n.

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- **Euler's theorem:** If $a \in \mathbb{Z}_n^*$, then

$$a^{\phi(n)} \equiv 1 \ mod \ n.$$

Pseudoprime

Definition

If n is an odd composite number and b is an integer s/t $\gcd(n, b) = 1$ and $b^{n-1} \equiv 1 \mod n$ then n is called a ***pseudoprime*** *to the base b. The integer b is called a* ***Fermat liar*** *(to primality) for n.*

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Example

1. *The number $n = 91$ is a pseudoprime to the base $b = 3$,*

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2. *However, 91 is not a pseudoprime to the base 2,*
$$\because 2^{90} \equiv 64 \mod 91.$$

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$$\because 3^{90} \equiv 1 \mod 91.$$
2. *However, 91 is not a pseudoprime to the base 2,* $\because 2^{90} \equiv 64 \mod 91.$
3. *The composite integer $n = 341 (= 11 \times 31)$ is a pseudoprime to the base 2,* $\because 2^{340} \equiv 1 \mod 341$.

Carmichael Number

Definition

A ***Carmichael number*** *is a composite integer n s/t*

$$b^{n-1} \equiv 1 \mod n,$$

for every $b \in \mathbb{Z}_n^*$.

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1. $n = 561 = 3 \times 11 \times 17$ is a Carmichael number. This is the smallest Carmichael number.

Carmichael Number

Definition

A *Carmichael number* *is a composite integer* n *s/t*

$$b^{n-1} \equiv 1 \mod n,$$

for every $b \in \mathbb{Z}_n^*$.

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2. The following are Carmichael numbers:
   - (a) $1105 = 5 \times 13 \times 17$
   - (b) $1729 = 7 \times 13 \times 19$
   - (c) $2465 = 5 \times 17 \times 29$

Carmichael Number

- A composite integer n is a Carmichael number iff the following two conditions are satisfied:
  - (i) n is square-free, and
  - (ii) $p - 1$ divides $n - 1$ for every prime divisor p of n.

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- A Carmichael number must be the product of at least three distinct primes.
- There are an infinite number of Carmichael numbers.

Quadratic Residue

Definition

Let $a \in \mathbb{Z}_n^*$; a is said to be a ***quadratic residue*** *modulo n, if*

$$\exists\, x \in \mathbb{Z}_n^* \ni x^2 \equiv a \mod n.$$

If no such x exists, then a is called a ***quadratic non-residue*** *modulo n.*

The set of all quadratic residues modulo n is denoted by Q_n and the set of all quadratic non-residues is denoted by $\overline{Q_n}$.

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Example

$\alpha = 6$ is a generator of $\mathbb{Z}_{13}^*$. The powers of α are

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$\alpha^i \mod 13$	1	6	10	8	9	2	12	7	3	5	4	11

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- Let $n = p.q$ be a product of two distinct odd primes. Then $a \in \mathbb{Z}_n^*$ is a quadratic residue modulo $n \Leftrightarrow a \in Q_p \;\&\; a \in Q_q$.
- It follows that $\#Q_n = \frac{(p-1)(q-1)}{4}$ and $\#\overline{Q_n} = \frac{3(p-1)(q-1)}{4}$.

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  Let $n = 21$.
  Then $Q_{21} = \{1, 4, 16\}$ and $\overline{Q_{21}} = \{2, 5, 8, 10, 11, 13, 17, 19, 20\}$.

The Legendre and Jacobi Symbols

- Let p be an odd prime and a an integer. The **Legendre symbol** $\left(\frac{a}{p}\right)$ is defined to be

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a, \\ 1, & \text{if } a \in Q_p, \\ -1, & \text{if } a \in \overline{Q_p}. \end{cases}$$

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- Let $n \geq 3$ be odd with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Then the **Jacobi symbol** $\left(\frac{a}{n}\right)$ is defined to be

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{e_1} \left(\frac{a}{p_2}\right)^{e_2} \cdots \left(\frac{a}{p_k}\right)^{e_k}$$

Properties of Legendre Symbol

(i) $\left(\frac{a}{p}\right) = a^{(p-1)/2} \mod p$. In particular, $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$. Hence, $-1 \in Q_p$ if $p \equiv 1 \mod 4$, and $-1 \in \overline{Q_p}$ if $p \equiv 3 \mod 4$.

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(iii) If $a \equiv b \mod p$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(iv) **Law of quadratic reciprocity:** If q is an odd prime distinct from p, then

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{(p-1)(q-1)/4}.$$

Fermat Test for Primality – Probabilistic Algorithm

Fermat Test for Primality

Input: n
Output: YES if n is composite, NO otherwise.
Choose a random $b,\ 0 < b < n$
if $\gcd(b, n) > 1$ **then**
  return YES
end
else ;
if $b^{n-1} \not\equiv 1 \mod n$ **then**
  return YES
end
else ;
return NO

The Euler Test – Probabilistic Algorithm

- If n is an odd prime, we know that an integer can have at most two square roots, $\mod n$. In particular, the only square roots of $1 \mod n$ are ± 1.
- If $a \not\equiv 0 \mod n$, $a^{(n-1)/2}$ is a square root of $a^{n-1} \equiv 1 \mod n$, so $a^{(n-1)/2} \equiv \pm 1 \mod n$.

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- If $a^{(n-1)/2} \not\equiv \pm 1 \mod n$ for some a with $a \not\equiv 0 \mod n$, then n is composite.

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- For a randomly chosen a with $a \not\equiv 0 \mod n$, compute $a^{(n-1)/2} \mod n$.

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  If n is large and chosen at random, the probability that n is prime is very close to 1.

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The Euler test is more powerful than the Fermat test.

The Euler Test – Probabilistic Algorithm

The Euler test is more powerful than the Fermat test.

- If the Fermat test finds that n is composite, so does the Euler test.
- If n is an odd composite integer (other than a prime power), 1 has at least 4 square roots $\mod n$.

  So we can have $a^{(n-1)/2} \equiv \beta \mod n$, where $\beta \neq \pm 1$ is a square root of 1.

  Then $a^{n-1} \equiv 1 \mod n$. In this situation, the Fermat Test (incorrectly) declares n a probable prime, but the Euler test (correctly) declares n composite.

Miller-Rabin Test – Probabilistic Algorithm

- The Euler test improves upon the Fermat test by taking advantage of the fact, if 1 has a square root other than $\pm 1 \mod n$, then n must be composite.
- If $a^{(n-1)/2} \not\equiv \pm 1 \mod n$, where $\gcd(a, n) = 1$, then n must be composite for one of two reasons:
    1. If $a^{n-1} \not\equiv 1 \mod n$, then n must be composite by Fermat's Little Theorem
    2. If $a^{n-1} \equiv 1 \mod n$, then n must be composite because $a^{(n-1)/2}$ is a square root of $1 \mod n$ different from ± 1.

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    - (i) If $a^{n-1} \not\equiv 1 \mod n$, then n must be composite by Fermat's Little Theorem
    - (ii) If $a^{n-1} \equiv 1 \mod n$, then n must be composite because $a^{(n-1)/2}$ is a square root of $1 \mod n$ different from ± 1.
- The limitation of the Euler test is that is does not go to any special effort to find square roots of 1, different from ± 1. The Miller-Rabin test does this.

Miller-Rabin Test – Probabilistic Algorithm

Miller-Rabin Test

Input: an odd integer $n \geq 3$ and security parameter $t \geq 1$.
Output: an answer "prime" or "composite" to the question: "Is n prime?"
Write $n - 1 = 2^s.r$ s/t r is odd.

```
for i = 1 to t do
    Choose a random integer a s/t 2 ≤ a ≤ n − 2.
    Compute y ≡ a^r mod n
    if y ≠ 1 & y ≠ n − 1 then
        j ← 1.
        while j ≤ s − 1 & y ≠ n − 1 do
            Compute y ← y^2 mod n.
            If y = 1 then return("composite").
            j ← j + 1.
        end
        If y ≠ n − 1 then return ("composite").
    end
end
Return("prime").
```

Deterministic Polynomial Time Algorithm

The AKS Algorithm

Input: a positive integer $n > 1$
Output: n is **Prime** or **Composite** in deterministic polynomial-time
If $n = a^b$ with $a \in \mathbb{N}$ & $b > 1$, then output **COMPOSITE.**

Deterministic Polynomial Time Algorithm

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If $n = a^b$ with $a \in \mathbb{N}$ & $b > 1$, then output **COMPOSITE.**
Find the smallest r such that $ord_r(n) > 4(\log n)^2$.
If $1 < \gcd(a, n) < n$ for some $a \leq r$, then output **COMPOSITE**.

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If $n \leq r$, then output **PRIME**.

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If $n \leq r$, then output **PRIME.**

for $a = 1$ *to* $\lfloor 2\sqrt{\phi(r)}\log n \rfloor$ **do**

  if $(x - a)^n \not\equiv (x^n - a) \mod (x^r - 1, n)$,

  then output **COMPOSITE.**

end

Return("PRIME").

RSA Example

- Suppose A wants to send the following message to B

  RSAISTHEKEYTOPUBLICKEYCRYPTOGRAPHY
- B chooses his $n = 737 = 11 \times 67$. Then $\phi(n) = 660$. Suppose he picks $e = 7, \Rightarrow d = 283$.
- $\because \; 26^2 < n < 26^3 \quad \therefore$ the block size of the plaintext $= 2$.

$$m_1 = `RS' = 17 \times 26 + 18 = 460$$

$$c_1 = 460^7 \equiv 697 \mod 737 = 1.26^2 + 0.26 + 21 = BAV$$

RSA Example

	RS	AI	ST	HE	KE	YT	OP	UB
m_b	460	8	487	186	264	643	379	521
c_b	697	387	229	340	165	223	586	5

LI	CK	EY	CR	YP	TO	GR	AP	HY
294	62	128	69	639	508	173	15	206
189	600	325	262	100	689	354	665	673

RSA Example

- Suppose A wants to send the following message to B

power

- B chooses his $n = 1943 = 29 \times 67$. Then $\phi(n) = 1848$. Suppose he picks $e = 701, \Rightarrow d = 29$.
- $\because\ 26^2 < n < 26^3 \quad \therefore$ the block size of the plaintext $= 2$.
- $m_1 = \text{`}po\text{'} = 15 \times 26 + 14 = 404,\ m_2 = \text{`}we\text{'} = 22 \times 26 + 4 = 576,\ m_3 = \text{`}ra\text{'} = 17 \times 26 + 0 = 442$.
- $c_1 = 404^{701} \equiv 1419 \mod 1943 = 2.26^2 + 2.26 + 15 = ccp$.
- $\|ly,\ c_2 = 344 = 13.26 + 6 = ang\ \&\ c_3 = 210 = 8.26 + 2 = aic$.
- The cipher text is

ccpangaic

Security of RSA

Security

If we know n and $\phi(n)$, we can find p & q.

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We have

$$\phi(n) = pq - p - q + 1 = n - (p + q) + 1.$$

Since we know n, we can find $p + q$ from the above equation. Since we know $pq = n$ and $p + q$, we can find p & q by factoring the quadratic equation

$$x^2 - (p + q)x + pq = 0.$$

Security of RSA

- Security of RSA relies on difficulty of finding d given n & e.
- Breaking RSA is no harder than Factoring.
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- RSA is secure against chosen plaintext attack (CPA).

IND-CCA

Security notion for encryption.

- From a ciphertext c, an attacker should not be able to derive any information from the corresponding plaintext m.
- Even if the attacker can obtain the decryption of any ciphertext, c excepted.
- This is called indistinguishability against a chosen ciphertext attack (IND-CCA).

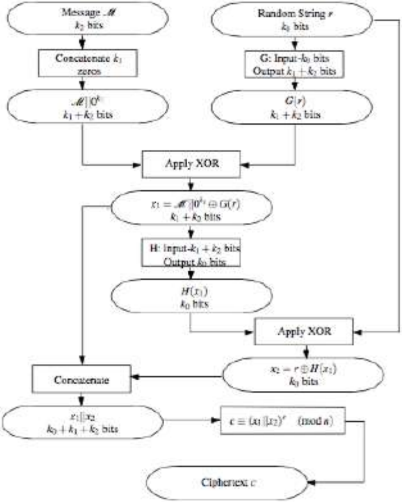
Choice of Encryption Key e

- The encryption exponent e should not be too small.

Choice of Encryption Key e

- The encryption exponent e should not be too small.
- Suppose $e = 3$ and there are 3 recipients having the same encryption exponent 3, but with different modulus $n_i, \quad i = 1, 2, 3$.
- Then, ciphertexts $y_i \equiv M^3 \mod n_i$ for $i = 1, 2, 3$ and send them to the recipients.
- Suppose two of them, say n_1 & n_2, are not coprime. Then, $\gcd(n_1, n_2)$ is a non-trivial factor of n_1 & n_2 and any adversary can factorise both of them.
- So, we can always assume that n_i for $i = 1, 2, 3$ are pairwise coprime.
- If adversary gets hold of the messages $y_i, \ 1 \leq i \leq 3$, (s)he can compute $M^3 \mod n_1n_2n_3$ using Chinese remainder theorem since $\gcd(n_i, n_j) = 1$ for $i \neq j$.
- Since $m < n_i, \ m^3 < n_1n_2n_3$. So, $M^3 \mod n_1n_2n_3 = M^3$ and the adversary can find M by taking the cube root of $M^3 \mod n_1n_2n_3$.

RSA in Practice – Optimal Asymmetric Encryption Padding (OAEP)



Optimal Asymmetric Encryption Padding (OAEP) I

- To encrypt a message M of k_2-bit, first concatenates the message with 0^{k_1}.
- Expands the message to $M\|0^{k_1}$.
- After that, select a random string r of length k_0 bits.
- Use it as the random seed for $G(r)$ and computes

$$x_1 = (M\|0^{k_1}) \oplus G(r), \quad x_2 = r \oplus H(x_1)$$

- If $x_1\|x_2$ is a binary number bigger than n, Alice chooses another random string r and computes the new values of x_1 & x_2.
- If $G(r)$ produces fairly random outputs, $x_1\|x_2$ will be less than n in binary with a probability greater than $\frac{1}{2}$.

Optimal Asymmetric Encryption Padding (OAEP) II

- After getting a string r with $x_1 \| x_2 < n$, Alice then encrypts $x_1 \| x_2$ to get the ciphertext

$$E(M) = (x_1 \| x_2)^e \equiv c \mod n$$

ElGamal PKC in $\mathbb{Z}_p^*$

Key Generation:

- $<\alpha>= \mathbb{Z}_p^*$, $\mathcal{P} = \mathbb{Z}_p^*$ & $\mathcal{C} = \mathbb{Z}_p^* \times \mathbb{Z}_p^*$.
- $\beta \equiv \alpha^a \mod p$.
- Public : p, α, β and Private : a.

Encryption:

- Select a random $k \in \mathbb{Z}_{p-1}$.
- $Enc_k(x) = (y_1, y_2)$

$$y_1 \equiv \alpha^k \mod p, \quad y_2 \equiv x.\beta^k \mod p.$$

Decryption:

$$Dec_k(y_1, y_2) \equiv y_2.(y_1^a)^{-1} \mod p.$$

ElGamal PKC in $\mathbb{Z}_p^*$

Example

- *Let* $p = 29$ *and* $\alpha = 2$, α *is a primitive element* $\mod 29$.
- *Let* $a = 5, \therefore \beta \equiv 2^5 \mod \equiv 3 \mod 29$.

ElGamal PKC in $\mathbb{Z}_p^*$

Example

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- *Let* $a = 5, \therefore \beta \equiv 2^5 \mod \equiv 3 \mod 29$.
- ***Public Key:*** $(29, 2, 3)$ *and* ***Private Key:*** 5
- *Plaintext:* $x = 6$ & *random number* $k = 14 \in \mathbb{Z}_{28}$

ElGamal PKC in $\mathbb{Z}_p^*$

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- *Let* $a = 5, \therefore \beta \equiv 2^5 \mod \equiv 3 \mod 29$.
- ***Public Key:*** $(29, 2, 3)$ *and* ***Private Key:*** 5
- *Plaintext:* $x = 6$ & *random number* $k = 14 \in \mathbb{Z}_{28}$
- $$y_1 \equiv 2^{14} \equiv 28 \mod 29 \;\&\; y_2 \equiv 6.3^{14} \equiv 23 \mod 29$$
- *Ciphertext: (28, 23).*

Elliptic Curves

- Elliptic curve[1] E over field $\mathbb{K}$ is defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{K}$$

- The set of $\mathbb{K}$-rational points $E(\mathbb{K})$ is defined as

$$E(\mathbb{K}) = \{(x, y) \in \mathbb{K} \times \mathbb{K} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{\mathcal{O}\}$$

Theorem

There exists an addition law on E and the set $E(K)$ with that addition forms a group.

[1] It is called a (generalized) Weierstrass equation. The equation defines a cubic curve called a Weierstrass curve.

Elliptic Curves I

1. Let $\mathbb{K}$ be a field of characteristic $\neq 2, 3$, and let $x^3 + ax + b$ be a cubic polynomial with no multiple roots ($-16(4a^3 + 27b^2) \neq 0 \Rightarrow 4a^3 + 27b^2 \neq 0$).
   An elliptic curve over $\mathbb{K}$ is the set of points (x, y) with $x, y \in K$ which satisfy the equation
   $$y^2 = x^3 + ax + b$$
   together with a single element denoted $\mathcal{O}$ and called the *point at infinity*.

Elliptic Curves II

2. If $char\ K = 2$, then an elliptic curve over $\mathbb{K}$ is the set of points satisfying an equation of type either

$$y^2 + cy = x^3 + ax + b$$

or

$$y^2 + xy = x^3 + ax + b$$

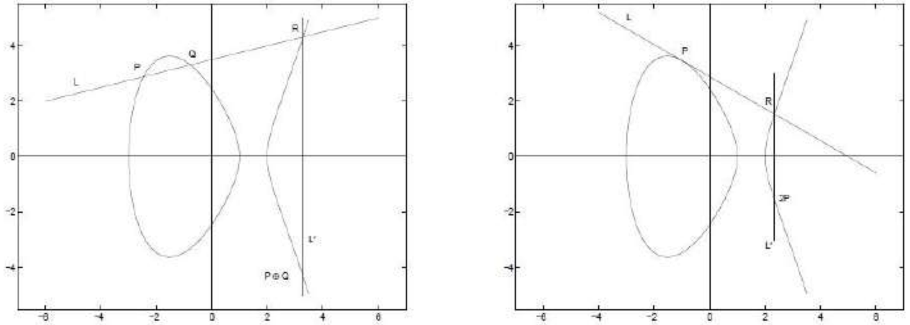
together with the *point at infinity* $\mathcal{O}$.

3. If $char\ K = 3$, then an elliptic curve over $\mathbb{K}$ is the set of points satisfying the equation

$$y^2 = x^3 + ax^2 + bx + c$$

together with the *point at infinity* $\mathcal{O}$.

Addition Law on Elliptic Curves



Adding two points

Doubling a point

$$y^2 = x^3 - 7x + 6$$

Addition Law on Elliptic Curves

- Suppose E is a non-singular elliptic curve.
- The point at infinity $\mathcal{O}$, will be the identity element, so $P + \mathcal{O} = \mathcal{O} + P = P \ \forall \ P \in E$.
- Suppose $P, Q \in E$, where $P = (x_1, y_1)$ & $Q = (x_2, y_2)$

(i) $x_1 \neq x_2$

- L is the line through P and Q.
- L intersects E in the two points P and Q
- L will intersect E in one further point R'.
- If we reflect R' in the x-axis, then we get a point R.

$$P + Q = R.$$

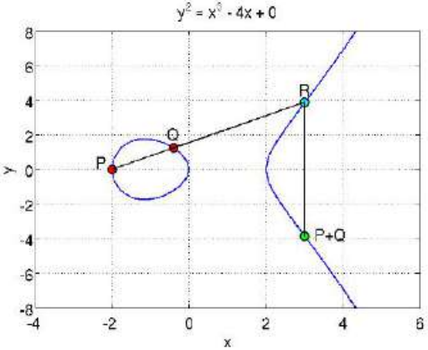
(ii) $x_1 = x_2$ & $y_1 = -y_2$

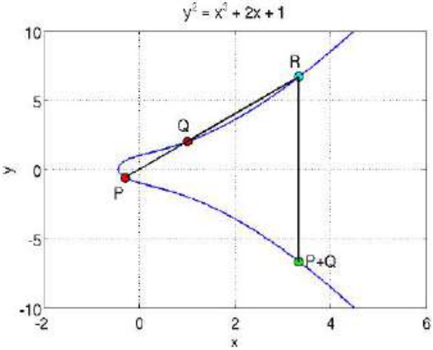
$$(x, y) + (x, -y) = O$$

(iii) $x_1 = x_2$ & $y_1 = y_2$

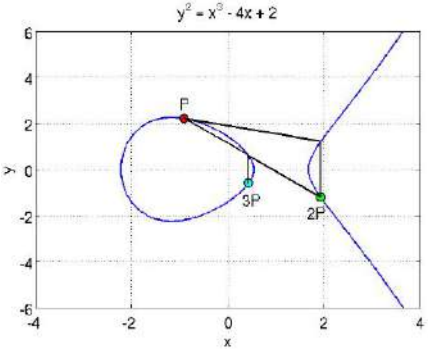
- Draw a tangent line L through P
- Follow step (i)

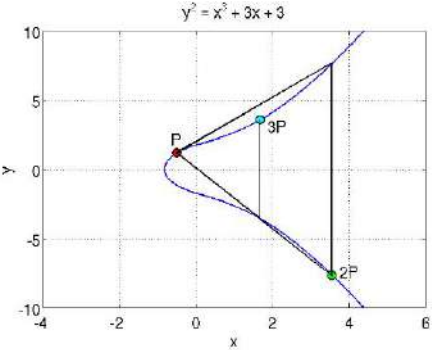
Addition Law on Elliptic Curves





Addition Law on Elliptic Curves





Addition Law on Elliptic Curves

- Suppose that we want to add the points $P_1 = (x_1, y_1)$ & $P_2 = (x_2, y_2)$ on the elliptic curve

$$E \; : \; y^2 = x^3 + ax + b.$$

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$$\lambda = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\ \frac{3x_1^2 + a}{2y_1} & \text{if } P_1 = P_2 \end{cases} \quad \text{and} \quad \nu = y_1 - \lambda x_1$$

Addition Law on Elliptic Curves

- Thus, we have

$$P_1 + P_2 = (x_3, -y_3),$$

where $x_3 = \lambda^2 - x_1 - x_2$ and $y_3 = \lambda x_3 + \nu$.

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- If $P_1 \neq P_2$ and $x_1 = x_2$, then $P_1 + P_2 = \mathcal{O}$.
- If $P_1 = P_2$ and $y_1 = 0$, then $P_1 + P_2 = 2P_1 = \mathcal{O}$.

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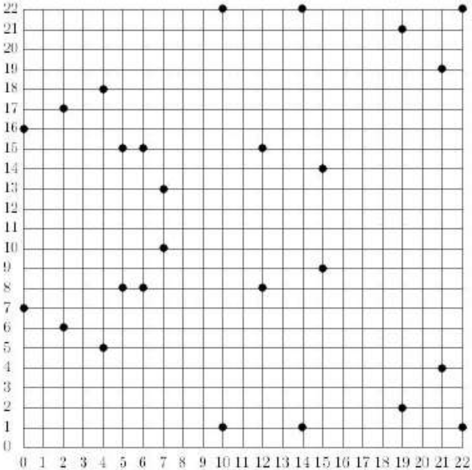
Visualizing Elliptic Curve Cryptography

Elliptic Curves over Finite Fields



The elliptic curve $y^2 = x^3 + x + 3 \bmod 23$

Elliptic Curves over Finite Fields

Problem

Let E be the elliptic curve $y^2 = x^3 + x + 1$ over $\mathbb{F}_{11}$. Then write down all the points of E over $\mathbb{F}_{11}$. Draw the elliptic curve E along with the grid.

Elliptic Curves over Finite Fields

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Let E be the elliptic curve $y^2 = x^3 + x + 1$ over $\mathbb{F}_{11}$. Then write down all the points of E over $\mathbb{F}_{11}$. Draw the elliptic curve E along with the grid.

Elliptic Curves over Finite Fields

Solution

- *First compute square of all the elements of* $\mathbb{F}_{11}$*:*

$$1^2 = 1,\ 2^2 = 4,\ 3^2 = 9,\ 4^2 = 5,\ 5^2 = 3,\ 6^2 = 3,\ 7^2 = 5,\ 8^2 = 9,\ 9^2 = 4,\ 10^2 = 1$$

Elliptic Curves over Finite Fields

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$Q_{11} = \{1, 3, 4, 5, 9\}$
$x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$
$x = 1 \Rightarrow y^2 = 3 \Rightarrow y = 5 \ or \ 6$
$x = 2 \Rightarrow y^2 = 0 \Rightarrow y = 0$
$x = 3 \Rightarrow y^2 = 9 \Rightarrow y = 3 \ or \ 8$
$x = 4 \Rightarrow y^2 = 3 \Rightarrow y = 5 \ or \ 6$
$x = 5 \Rightarrow y^2 = 10$
$x = 6 \Rightarrow y^2 = 3 \Rightarrow y = 5 \ or \ 6$
$x = 7 \Rightarrow y^2 = 10$
$x = 8 \Rightarrow y^2 = 4 \Rightarrow y = 2 \ or \ 9$
$x = 9 \Rightarrow y^2 = 2$
$x = 10 \Rightarrow y^2 = 10$
$E(\mathbb{F}_{11}) = \{\mathcal{O}, (0, 1), (0, 10), (1, 5), (1, 6), (2, 0), (3, 3), (3, 8), (4, 5), (4, 6), (6, 5), (6, 6), (8, 2), (8, 9)\}$

NIST's Primes for ECC

$$
\begin{aligned}
p_{192} &= 2^{192} - 2^{64} - 1 \\
p_{224} &= 2^{224} - 2^{96} + 1 \\
p_{256} &= 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1 \\
p_{384} &= 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1 \\
p_{521} &= 2^{521} - 1
\end{aligned}
$$

$$
\begin{aligned}
W - 25519 &= 2^{255} - 19 \\
W - 448 &= 2^{448} - 2^{224} - 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Edwards}25519 &= 2^{255} - 19 \\
\text{Edwards}448 &= 2^{448} - 2^{224} - 1
\end{aligned}
$$

Recommendations for Discrete Logarithm-Based Cryptography: Elliptic Curve Domain Parameters

ElGamal Cryptosystems on Elliptic Curves

- First choose two public elliptic curve points P and Q s/t

$$Q = sP,$$

  where s is the private key.
- To encrypt choose a random k
- $Enc_k(m) = (y_1, y_2)$

$$y_1 = kP, \quad y_2 = m + kQ.$$

- **Decryption:**

$$Dec_k(y_1, y_2) = y_2 - s.y_1$$

ElGamal Cryptosystems on Elliptic Curves

- The plaintext space in general may not consist of the points on the curve E.
- So, we convert the plaintext as an arbitrary element in $\mathbb{Z}_p$.
- After that, we can apply a suitable hash function $h : E \to \mathbb{Z}_p$ is applied to kQ
- To encrypt a messaxe m choose a random k
- The ciphertext $c = Enc_k(m) = (y_1, y_2)$

$$y_1 = kP, \quad y_2 \equiv (m + h(kQ)) \mod p.$$

- **Decryption:**

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$$y_1 = kP, \quad y_2 \equiv (m + h(kQ)) \mod p.$$

- **Decryption:**
  - Compute $h(kQ)$
  - Compute $c \equiv (y_2 - h(kQ)) \mod p$

ElGamal Cryptosystems on Elliptic Curves

Key Generation

- Let E be an elliptic curve defined over $\mathbb{Z}_p$ (where $p > 3$ is prime) s/t E contains a cyclic subgroup $H = \langle P \rangle$ of prime order n in which the **Discrete Logarithm Problem** is infeasible.
- Let $h : E \to \mathbb{Z}_p$ be a secure hash function.
- Let $\mathcal{P} = \mathbb{Z}_p$ and $\mathcal{C} = (\mathbb{Z}_p \times \mathbb{Z}_2) \times \mathbb{Z}_p$. Define

$$\mathcal{K} = \{(E, P, s, Q, n, h) \ : \ Q = sP\},$$

where P and Q are points on E and $s \in \mathbb{Z}_n^*$. The values $E, P, Q, n,$ and h are the public key and s is the private key.

ElGamal Cryptosystems on Elliptic Curves

Encryption

- To encrypt a message m sender selects a random number $k \in \mathbb{Z}_n^*$ and compute the ciphertext

$$y = e_K(m, k) = (y_1, y_2) = (\text{POINT-COMPRESS}(kP), m + h(kQ) \mod p),$$

where $y_1 \in \mathbb{Z}_p \times \mathbb{Z}_2$ and $y_2 \in \mathbb{Z}_p$.

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where $y_1 \in \mathbb{Z}_p \times \mathbb{Z}_2$ and $y_2 \in \mathbb{Z}_p$.

Decryption

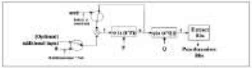
$$d_K(y) = y_2 - h(R) \mod p,$$

where $R = s\text{POINT-DECOMPRESS}(y_1)$.

The Many Flaws of Dual_EC_DRBG

Matthew Green in Dual EC, NSA, RNGs · September 18, 2013 · 3,055 Words

The Many Flaws of Dual_EC_DRBG



The Dual_EC_DRBG generator from NIST SP800-90A.

Update 9/19: *RSA warns developers not to use the default Dual_EC_DRBG generator in BSAFE. Oh lord.*

As a technical follow up to my previous post about the NSA's war on crypto, I wanted to make a few specific points about standards. In particular I wanted to address the allegation that NSA inserted a backdoor into the Dual-EC pseudorandom number generator.

For those not following the story, Dual-EC is a pseudorandom number generator proposed by NIST for international use back in 2006. Just a few months later, Shumow and Ferguson made cryptographic history by pointing out that there might be an NSA backdoor in the algorithm. This possibility — fairly remarkable for an algorithm of this type — looked bad and smelled worse. If true, it spelled almost certain doom for anyone relying on Dual-EC to keep their system safe from spying eyes.

Key Comparison

Symmetric Key Size (in bits)	Based on Factoring (in bits)	Based on DLP (in bits)	Based on ECDLP (in bits)
80	1024	1024	160
112	2048	2048	224
128	3072	3072	256
192	7680	7680	384
256	15360	15360	512

Outline

Signature Scheme

Definition

A signature scheme is a five-tuple $(\mathcal{P}, \mathcal{A}, \mathcal{K}, \mathcal{S}, \mathcal{V})$, where the following conditions are satisfied:

(i) $\mathcal{P}$ is a finite set of possible messages

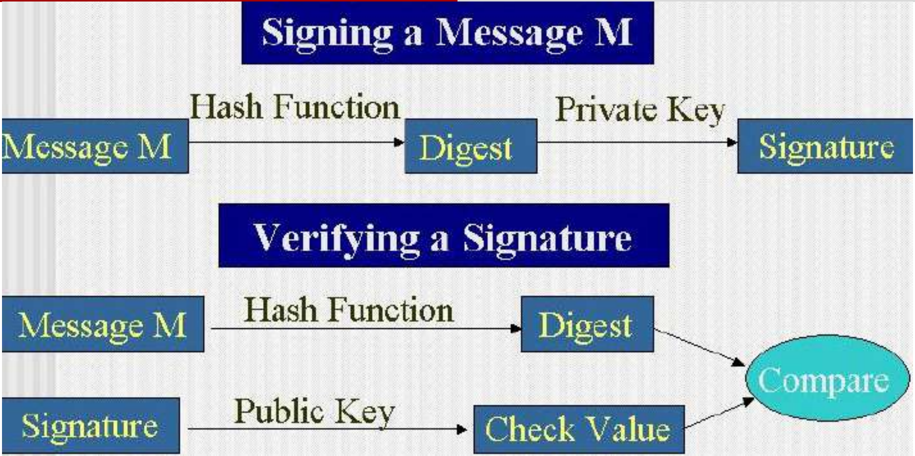
(ii) $\mathcal{A}$ is a finite set of possible signatures

(iii) $\mathcal{K}$, the keyspace, is a finite set of possible keys

(iv) For each $K \in \mathcal{K}$, there is a signing algorithm $sig_K \in \mathcal{S}$ and a corresponding verification algorithm $ver_K \in \mathcal{V}$. Each $sig_K : \mathcal{P} \to \mathcal{A}$ and $ver_K : \mathcal{P} \times \mathcal{A} \to \{true, \ false\}$ are functions s/t the following equation is satisfied for every message $x \in \mathcal{P}$ and for every signature $y \in \mathcal{A}$

$$ver_K = \begin{cases} \text{true} & \text{if } y = sig_K(x) \\ \text{false} & \text{if } y \neq sig_K(x) \end{cases}$$

A pair (x, y) with $x \in \mathcal{P}$ and $y \in \mathcal{A}$ is called a signed message.


Signing a Message M
Message M
Hash Function
Digest
Private Key
Signature
Verifying a Signature
Message M
Hash Function
Digest
Compare
Signature
Public Key
Check Value

RSA Signature Scheme

Signature Generation

A signs a message m. Any entity B can verify A's signature and recover the message m from the signature.

- Compute $\tilde{m} = R(m)$, where $R : \mathcal{M} \to \mathbb{Z}_n$.
- Compute $s \equiv \tilde{m}^d \mod n$.
- A's signature for m is s.

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Signature Verification

To verify A's signature s and recover the message m, B should:

- Obtain A's authentic public key (n, e).
- Compute $\tilde{m} \equiv s^e \mod n$.
- Verify that $\tilde{m} \in$ range of $\mathcal{M}$; if not, reject the signature.
- Recover $m = R^{-1}(\tilde{m})$.

DSA

Key Generation

1. Choose a hash function h.
2. Decide a key length L.
3. Choose prime q with with same number of bits as output of h.
4. Choose α-bit prime p such that $q|(p-1)$.
5. Choose g such that $g^q \equiv 1 \mod p$.

Choose x : $0 < x < q$.
Calculate : $y \equiv g^x \mod p$.
(p, q, g, y) $\longrightarrow$ Public Key
x $\longrightarrow$ Private Key

DSA

Signature Generation

1. Generate random k such that $0 < k < q$.
2. Calculate $r \equiv (g^k \mod p) \mod q$.
3. Calculate $s \equiv (k^{-1}(h(m) + xr)) \mod q$.
4. Signature is (r, s).

DSA

Signature Generation

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Signature Verification

1. $w \equiv s^{-1} \mod q$.
2. $u_1 \equiv (h(m).w) \mod q$.
3. $u_2 \equiv rw \mod q$.
4. $v \equiv (g^{u_1}.y^{u_2} \mod p) \mod q$.
5. Verify $v = r$.

Schnorr Signature Scheme

Key Generation

- Let p be a prime s/t the DLP in $\mathbb{Z}_p^*$ is intractable, and let q be a prime and $q \mid (p-1)$. Let $\alpha \in \mathbb{Z}_p^*$ be a q^{th} root of unity modulo p. Let $\mathcal{P} = \{0,1\}^*$, $\mathcal{A} = \mathbb{Z}_q \times \mathbb{Z}_q$, and define

$$\mathcal{K} = \{(p, q, \alpha, a, \beta) \ : \ \beta \equiv \alpha^a \mod p\},$$

  where $0 \leq a \leq q-1$.

  The values p, q, α, and β are the public key, and a is the private key.

  Finally, let $h \ : \ \{0,1\}^* \to \mathbb{Z}_q$ be a secure hash function.

Schnorr Signature Scheme

Signature Generation

- Signer first selects a (secret) random number $k,\ 1 \leq k \leq q-1$, define

$$sig_K(x,k) = (\gamma, \delta),$$

where

$$\gamma = h(x \| \alpha^k \ mod\ p)\ \&\ \delta = k + a\gamma \ mod\ q.$$

Verification

- For $x \in \{0,1\}^*$ and $\gamma, \delta \in \mathbb{Z}_q$, verification is done by performing the following computations:

$$ver_K(x,(\gamma,\delta)) = true \iff h(x \| \alpha^{\delta}\beta^{-\gamma} \ mod\ p) = \gamma.$$

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Thanks a lot for your attention!