Introduction to Graph Theory

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1

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Outline

Outline

Introduction

- Graph theory is a practical branch of mathematics that deals with the arrangements of certain objects known as vertices (or nodes) and the relationships between them.

[1] The 4-colour theorem states that given any map it is possible to colour the regions of the map with no more than four colours such that no two adjacent regions have the same colour.

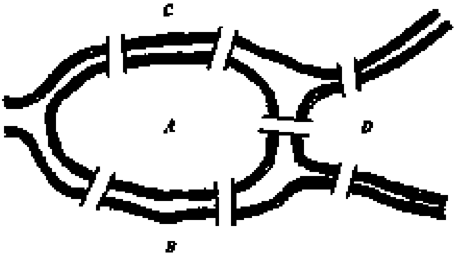
Introduction

- Graph theory is a practical branch of mathematics that deals with the arrangements of certain objects known as vertices (or nodes) and the relationships between them.
- It has been applied to practical problems such as the modelling of computer networks, determining the shortest driving route between two cities, the link structure of a website, the travelling salesman problem, electric networks, organic chemical isomers, and the four-colour problem[1].

[1] The 4-colour theorem states that given any map it is possible to colour the regions of the map with no more than four colours such that no two adjacent regions have the same colour.

Discovery

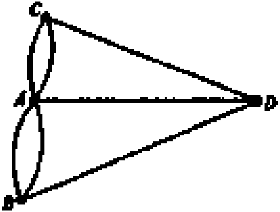
Problem (Königsberg Bridge Problem)



Begin at any of the four land areas $A, B, C, \ \& \ D$, walk across each bridge exactly once and return to the starting point.

Discovery

Problem (Königsberg Bridge Problem)



- Euler settled this famous unsolved problem in 1736
- He became the father of graph theory as well as topology
- Euler replaced each land area by a point and each bridge by a line joining the corresponding points, thereby producing a “graph"

Graphs

Definition

A graph $G = (V, E)$ consists of a ***nonempty*** *set V of vertices (or nodes) and a set $E(\subseteq V \times V)$ of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.*

- We write $V(G)$ for the set of vertices/nodes/points.
- We denote $E(G)$ for the set of edges/lines/arcs of a graph G.
- $|G| = |V(G)|$ denotes the number of vertices.
- $e(G) = |E(G)|$ denotes the number of edges.

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vertex	$\rightarrow$	point	node	junction	0-simplex
edge	$\rightarrow$	line	arc	branch	1-simplex

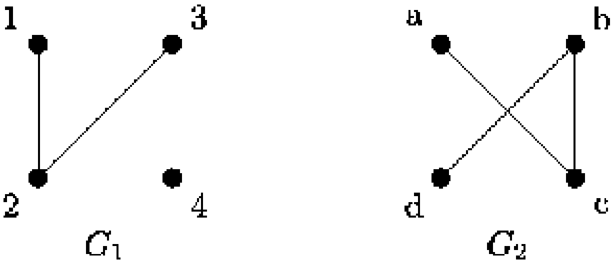
Graphs

Definition

- *A loop is an edge (v, v) for some $v \in V$.*
- *An edge $e = (u, v)$ is a multiple edge if it appears multiple times in E.*
- *A graph is simple if it has no loops or multiple edges.*
- *Multigraphs may have multiple edges connecting the same two vertices. When m different edges connect the vertices u & v, we say that $\{u, v\}$ is an edge of multiplicity m.*
- *A pseudograph may include loops, as well as multiple edges connecting the same pair of vertices.*

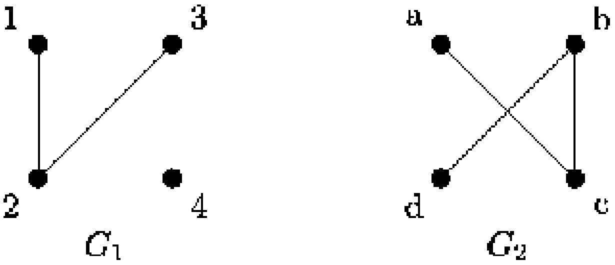
Graphs

Example (Example of Simple Graphs)



Graphs

Example (Example of Simple Graphs)



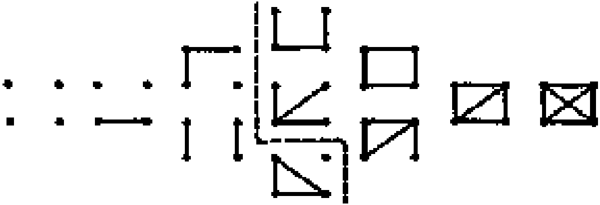
The function $\phi : G_1 \to G_2$ given by

$\phi(1) = a, \phi(2) = c, \phi(3) = b, \phi(4) = d$

is an isomorphism.

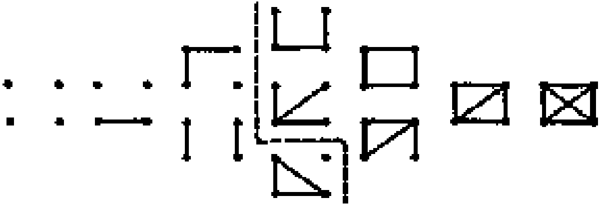
Graphs

Example (Varieties of Graphs)



Graphs

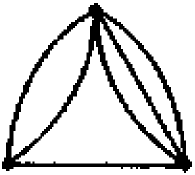
Example (Varieties of Graphs)

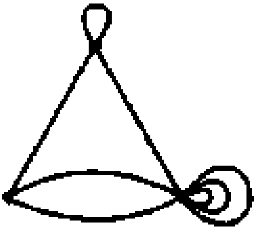


- every graph with four points is isomorphic with one of these
- the 5 graphs to the left of the dashed curve are disconnected
- the 6 graphs to its right are connected
- the last graph is complete
- the first graph is totally disconnected
- the first graph with four lines is a cycle
- the first graph with three lines is a path

Graphs

Example (Multigraph & Pseudograph)





Remark:

We have a lot of freedom when we draw a picture of a graph. All that matters is the connections made by the edges, not the particular geometry depicted. For example, the lengths of edges, whether edges cross, how vertices are depicted, and so on, do not matter

Graphs

Definition

- *Vertices* u, v *are adjacent in* G *if* $(u, v) \in E(G)$.
- *An edge* $e \in E(G)$ *is incident to a vertex* $v \in V(G)$ *if* $v \in e$.
- *If* $(u, v) \in E$ *then* v *is a neighbour of* u.

Graphs

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Remark:

- A graph with an infinite vertex set V is called an infinite graph. A graph with a finite vertex set is called a finite graph.
- We restrict our attention to finite graphs only.

Directed Graphs

Definition

*A directed graph or digraph D consists of a **finite nonempty** set V of points together with a prescribed collection E of ordered pairs of distinct points.*

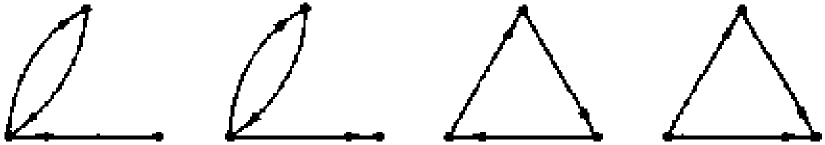
The elements of E are directed lines or arcs.

An oriented graph is a digraph having no symmetric pair of directed lines.

- Each edge is associated with an ordered pair of vertices.
- The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.
- By definition, a digraph has no loops or multiple arcs.

Directed Graphs

Example



- All digraphs with three points and three arcs are shown
- The last two are oriented graphs.

Directed Graphs

Definition

A graph G is labeled when the p points are distinguished from one another by names such as

$$v_1, v_2, \ldots, v_p.$$

Definition

A simple directed graph has no loops and no multiple edges.

A directed multigraph may have multiple directed edges. When there are m directed edges from the vertex u to the vertex v, we say that (u, v) is an edge of multiplicity m.

Graph Terminology: Summary

Type	Edges	Multiple Edges	Loops
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed & undirected	Yes	Yes

Applications of Graphs

- Computer Networks (vertices represent data centers & edges represent communication links.)
- Social networks (individuals/organizations are represented by vertices; relationships between individuals/organizations are represented by edges)
- Communications networks (vertices represent devices & edges represent the particular type of communications links of interest)
- Information networks
- Software design
- Transportation networks
- Biological networks
- ⋮

Outline

Basic Terminology

Definition

- *Let $G = (V, E)$ be a graph. Each pair $x = \{u, v\}$ of points in E is a line of G, and x is said to join u & v.*
- *We denote $x = uv$ and say that u & v are adjacent points (sometimes denoted by $u\ adj\ v$).*
- *Point u and line x are incident with each other (||ly for v & x).*
- *If two distinct lines x & y are incident with a common point, then they are called adjacent lines.*
- *A graph with p points and q lines is called a (p, q) graph[a].*

[a] The $(1, 0)$ graph is the trivial graph

Graph Isomorphism

Definition

Two (labeled) graphs G & H are isomorphic (denoted by $G \cong H$) if $\exists$ a one-to-one correspondence between their point sets which preserves adjacency.

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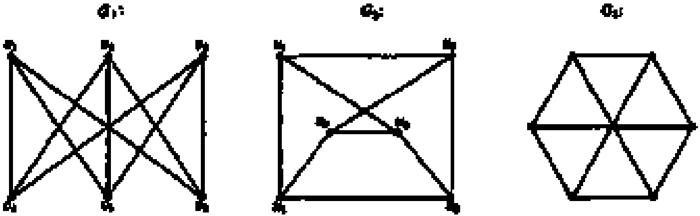

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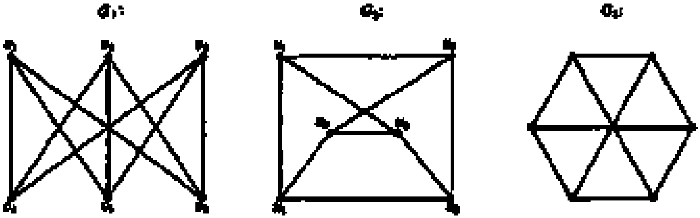


Isomorphism is an equivalence relation of graphs.

Graph Isomorphism

Definition

An invariant of a graph G is a number associated with G which has the same value for any graph isomorphic to G. Thus the numbers p & q are certainly invariants.

- A complete set of invariants determines a graph up to isomorphism.
- No decent complete set of invariants for a graph is known.

Subgraph

Definition

1. *A subgraph of G is a graph having all of its points and lines in G.*

   If G_1 is a subgraph of G, then G is a supergraph of G_1.

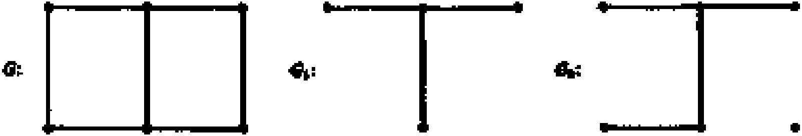
2. *A spanning subgraph is a subgraph containing all the points of G.*

3. *For any set S of points of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with point set S.*

   Thus two points of S are adjacent in $\langle S \rangle$ iff they are adjacent in G.

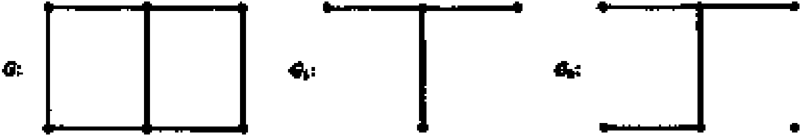
Subgraph

Example



Subgraph

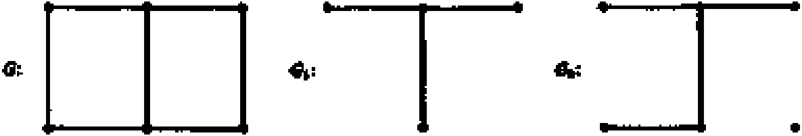
Example



- G_2 is a **spanning subgraph** of G but G_1 is not.

Subgraph

Example



- G_2 is a **spanning subgraph** of G but G_1 is not.
- G_1 is an **induced subgraph** but G_2 is not.

Subgraph

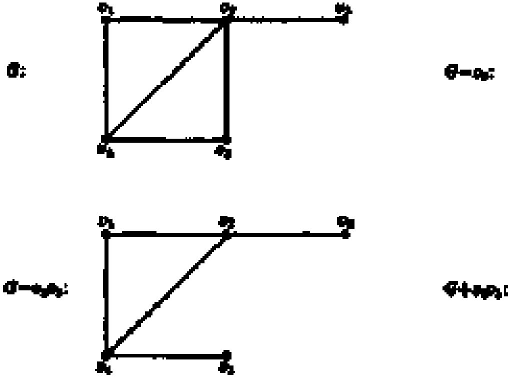


Figure: A graph ± a specific point or line

Adjacency Matrices

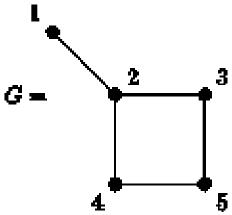
Definition

Let $G = (V, E)$ *be a graph with* $V = \{1, 2, \ldots, n\}$. *The* *adjacency matrix* $A = A(G)$ *is the* $n \times n$ *symmetric matrix defined by*

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

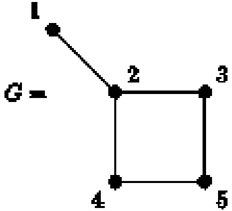
Adjacency Matrices

Example



Adjacency Matrices

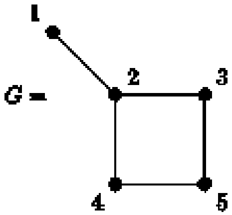
Example



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrices

Example



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Remark: Any adjacency matrix A is real and symmetric

Incidence Matrices

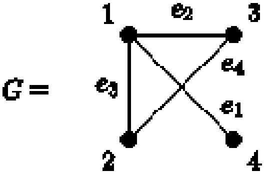
Definition

Let $G = (V, E)$ *be a graph with* $V = \{v_1, \ldots, v_n\}$ *and* $E = \{e_1, \ldots, e_m\}$*. Then the incidence matrix* $B = B(G)$ *of* G *is the* $n \times m$ *matrix defined by*

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j, \\ 0 & \text{otherwise.} \end{cases}$$

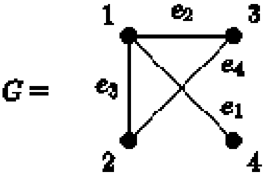
Incidence Matrices

Example



Incidence Matrices

Example

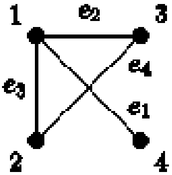


$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Incidence Matrices

Example

$G =$



$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Remark: Every column of B has 2 entries 1.

Degrees of Vertices

Definition

The degree[a] of a point $v_i \in G$, denoted by d_i or $deg\ v_i$, is the number of lines incident with v_i.

[a] sometimes called **valency**

- Every line is incident with two points, it contributes 2 to the sum of the degrees of the points.

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Theorem (Euler)

The sum of the degrees of the points of a graph G *is twice the number of lines*[a],

$$\sum deg\ v_i = 2q$$

[a] it was the first theorem of graph theory! Also called **Handshaking Theorem**

Degrees of Vertices

Exercise

1. *How many edges are there in a graph with 10 vertices each of degree 3?*
2. *If a graph has 5 vertices, can each vertex have degree 3?*

Degrees of Vertices

Exercise

1. *How many edges are there in a graph with 10 vertices each of degree 3?*
2. *If a graph has 5 vertices, can each vertex have degree 3?*

Corollary

In any graph, the number of points of odd degree is even.

Degrees of Vertices

Definition

1. *In a* (p, q) *graph,* $0 \leq deg\ v \leq p - 1$ *for every point* v.
   The minimum degree among the points of G *is denoted* $min\ deg\ G$ *or* $\delta(G)$

   while $\Delta(G) = max\ deg\ G$ *is the largest such number.*
2. *If* $\delta(G) = \Delta(G) = r$, *then all points have the same degree and* G *is called regular of degree* r.

   We then say of the degree of G *and write* $deg\ G = r$.

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Corollary

Every cubic graph has an even number of points.

Degrees of Vertices

Definition

The point v is said to be isolated if $deg\ v = 0$ and it is said to be an endpoint if $deg\ v = 1$.

Problem

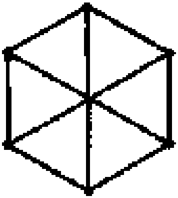
Prove that at any party with 6 people, there are 3 mutual acquaintances or 3 mutual non-acquaintances.

Complement of a Graph

Definition

1. *The complement* $\bar{G}$ *of a graph* G *has* $V(G)$ *as its point set, but two points are adjacent in* $\bar{G}$ *iff they are not adjacent in* G.
2. *A self complementary graph is isomorphic with its complement.*

G:



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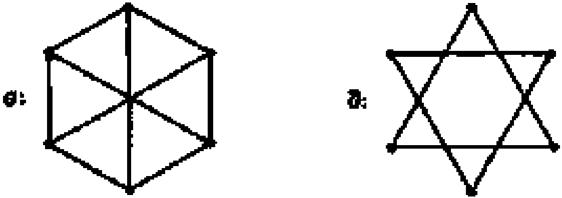

Figure: A graph and its complement

Self-complementary Graph

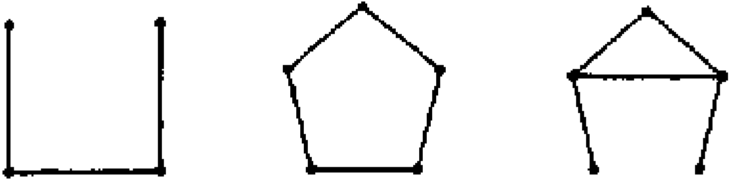

Figure: The smallest nontrivial self-complementary graphs

Special Graphs

Definition

The complete graph or clique[a] K_p *has every pair of its* p *points adjacent.*
Thus K_p *has* $\binom{p}{2}$ *lines and is regular of degree* $p-1$.

[a] complete subgraph

Definition

$G=(V,E)$ *is bipartite if there is a partition* $V=V_1\cup V_2$ *into two disjoint sets such that each* $e\in E(G)$ *intersects both* V_1 *and* V_2.

Definition

$K_{n,m}$ *is the complete bipartite graph. Take* $n+m$ *vertices partitioned into a set* A *of size* n *and a set* B *of size* m*, and include every possible edge between* A & B.

Degrees of Vertices

Theorem

For any graph G with 6 points, G or $\bar{G}$ contains a triangle.

Degrees of Vertices

Theorem

For any graph G with 6 points, G or $\bar{G}$ contains a triangle.

Proof.

- Let v be a point of the graph G.
- $\because v$ is adjacent either in G or in $\bar{G}$ to the other five points of G, so, we can assume without loss of generality that there are 3 points $u_1 u_2, u_3$ adjacent to v in G.
- If any 2 of these points are adjacent, then they are 2 points of a triangle whose third point is v.
- If no 2 of them are adjacent in G, then u_1, u_2, u_3 are the points of a triangle in $\bar{G}$.

□

Degrees of Vertices

Theorem

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□

Theorem

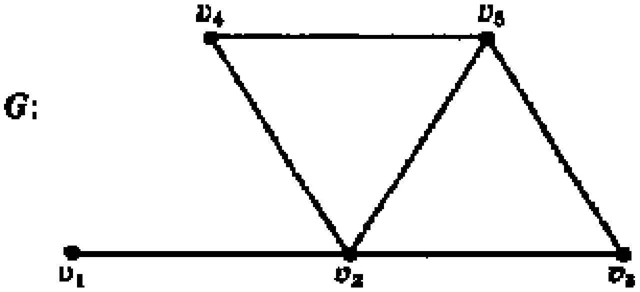
The maximum number of lines among all p point graphs with no triangles is $\lfloor p^2/4 \rfloor$.

Walk & Connectedness

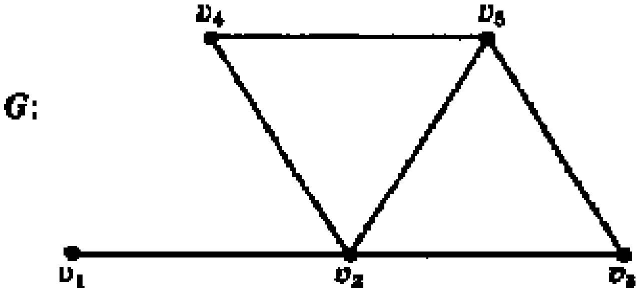
Definition

- *A walk of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$, beginning and ending with points, in which each line is incident with the two points immediately preceding and following it.*
- *This walk joins v_0 & v_n and may also be denoted as $v_0 v_1 v_2 \ldots v_n$ (the lines being evident by context); it is sometimes called a $v_0 - v_n$ walk. It is closed if $v_0 = v_n$ and is open otherwise.*
- *It is a trail if all the lines are distinct.*
- *It is a path if all the points (and thus necessarily all the lines) are distinct.*
- *A closed path is called a cycle/circuit provided its n $(n \geq 3)$ points are distinct.*
  We denote by C_n the graph consisting of a cycle with n points and by P_n a path with n points.
- *A wheel W_n is obtained by adding an additional vertex to a cycle C_n and connecting this new vertex to each of the n vertices in C_n by new edges.*

Walk & Connectedness



Walk & Connectedness



- $v_1 v_2 v_5 v_2 v_3$ is a walk which is not a trail
- $v_1 v_2 v_5 v_4 v_2 v_3$ is a trail which is not a path
- $v_1 v_2 v_5 v_4$ is a path and $v_2 v_4 v_5 v_2$ is a cycle.

Walk & Connectedness

Definition

A graph is connected if every pair of points are joined by a path.

A maximal connected subgraph of G is called a connected component or simply a component of G.

Thus, a disconnected graph has at least two components.

Proposition

Every walk from $u - v$ in G contains a path between u & v.

Proposition

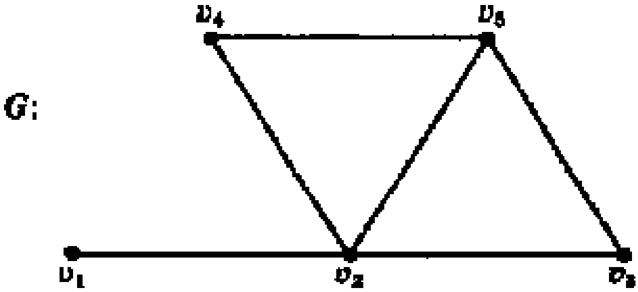
A graph with n vertices and m edges has at least $n - m$ connected components.

Walk & Connectedness

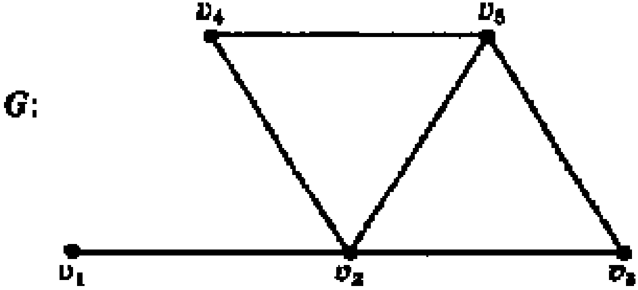
Definition

- *The length of a walk $v_0 \cdots v_n$ is n, the number of occurrences of lines in it.*
- *The girth of a graph G, denoted $g(G)$ is the length of a shortest cycle (if any) in G.*
- *The circumference of a graph G, $c(G)$ is the length of any longest cycle (if any).*
- *A shortest $u - v$ path is called a geodesic.*
- *The diameter $d(G)$ of a connected graph G is the length of any longest geodesic.*

Walk & Connectedness



Walk & Connectedness



- The graph G has *girth* $g = 3$, *circumference* $c = 4$, and *diameter* $d = 2$.

Walk & Connectedness

- The distance $d(u, v)$ between two points u & v in G is the length of a shortest path joining them (if any); otherwise $d(u, v) = \infty$.
- In a connected graph, distance is a *metric*;

Walk & Connectedness

- The distance $d(u, v)$ between two points u & v in G is the length of a shortest path joining them (if any); otherwise $d(u, v) = \infty$.
- In a connected graph, distance is a *metric*; i.e., for all points $u, v,$ & w,

$$d : V \times V \to \mathbb{R}$$

such that

(i) $d(u, v) \geq 0$, with $d(u, v) = 0$ iff $u = v$

(ii) $d(u, v) = d(v, u)$

(iii) $d(u, v) \leq d(u, w) + d(w, v)$

Walk & Connectedness

Theorem

A graph is bipartite iff all its cycles are even.

Walk & Connectedness

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Proof.

- If G is a bipartite, then its point set V can be partitioned into two sets V_1 & V_2 so that every line of G joins a point of V_1 with a point of V_2.
- Thus every cycle $v_1 v_2 \cdots v_n v_1$ in G necessarily has its oddly subscripted points in V_1 (say), and the others in V_2, so that its length n is even.

□

Walk & Connectedness

Theorem

A graph is bipartite iff all its cycles are even.

Proof.

- For the converse, now we assume, without loss of generality, that G is connected.
- Take any point $v_1 \in V$, and let $v_1 \in V_1$ and all points at even distance from v_1 while $V_2 = V \setminus V_1$.
- Since all the cycles of G are even, every line of G joins a point of V_1 with a point of V_2.
- For suppose there is a line uv joining 2 points of V_1. Then the union of geodesics from v_1 to v and from v_1 to u together with the line uv contains an odd cycle, a contradiction.

□

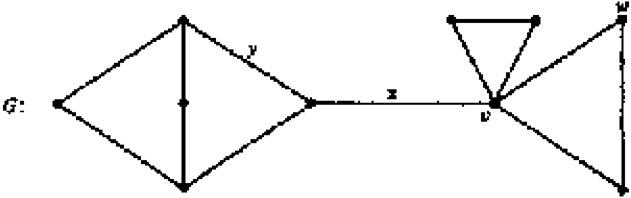
Block

Definition

- *A cutpoint of a graph is one whose removal increases the number of components.*

  Thus, if v is a cutpoint of a connected graph G, then $G \setminus \{v\}$ is disconnected.
- *A bridge is a line whose removal increases the number of components.*
- *A nonseparable graph is connected, nontrivial, and has no cutpoints.*
- *A block of a graph is a maximal nonseparable subgraph. If G is nonseparable, then G itself is often called a block.*

Block



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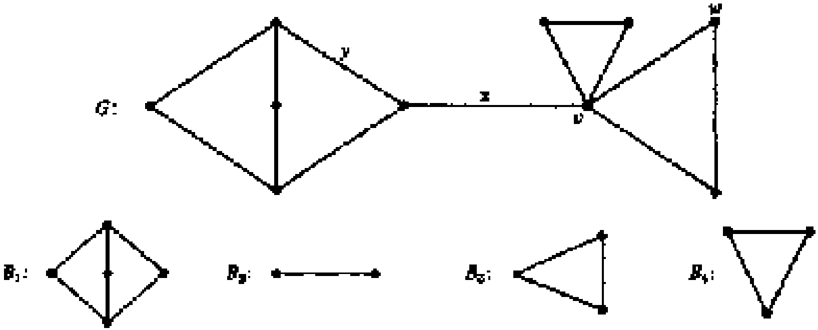

Figure: A graph and its blocks

Block

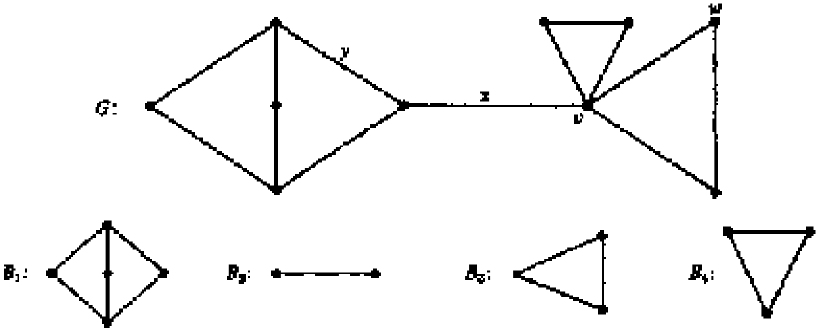

Figure: A graph and its blocks

v is a cutpoint while w is not and x is a bridge but y is not

Block

Theorem

Let v be a point of a connected graph G. The following statements are equivalent:

1. (i) v *is a cutpoint of* G.
2. (ii) *There exist points* u & w *distinct from* v *s/t* v *is on every* $u - w$ *path.*
3. (iii) *There exists a partition of the set of points* $V \setminus \{v\}$ *into subsets* U & W *s/t for any points* $u \in U$ *and* $w \in W$, *the point* v *is on every* $u - w$ *path.*

Block

Theorem

Let x be a line of a connected graph G. The following statements are equivalent:

(i) x *is a bridge of* G.

(ii) x *is not on any cycle of* G.

(iii) *There exist points* u & v *of* G *s/t the line* x *is on every path joining* u *and* v.

(iv) *There exists a partition of* V *into subsets* U & W *s/t for any points* $u \in U$ *and* $w \in W$, *the line* x *is on every path joining* u *and* w.

Block

Theorem

Let G be a connected graph with at least 3 points. The following statements are equivalent:

- (i) *G is a block.*
- (ii) *Every 2 points of G lie on a common cycle.*
- (iii) *Every point and line of G lie on a common cycle.*
- (iv) *Every 2 lines of G lie on a common cycle.*
- (v) *Given 2 points and one line of G, there is a path joining the points which contains the line.*
- (vi) *For every 3 distinct points of G, there is a path joining any 2 of them which contains the third.*
- (vii) *For every 3 distinct points of G, there is a path joining any two of them which does not contain the third.*

Eulerian Graphs

Definition

Given a graph G, if it is possible to find a walk that traverses each line exactly once, goes through all points, and ends at the starting point, we call G is Eulerian.

Thus, an Eulerian graph has an Eulerian trail a closed trail containing all points and lines.

Clearly, an Eulerian graph must be connected.

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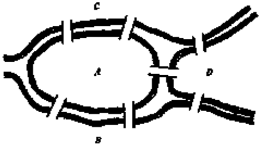
Problem (Königsberg Bridge Problem)



Begin at any of the four land areas $A, B, C,$ & D, walk across each bridge exactly once and return to the starting point.

Eulerian Graphs

Example

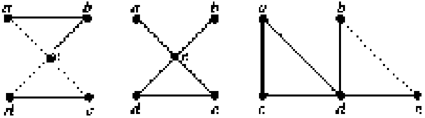


Figure: G_1 G_2 G_3

Eulerian Graphs

Example

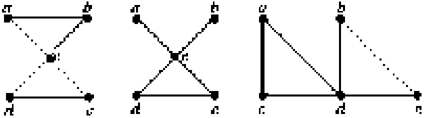


Figure: G_1 G_2 G_3

- *The graph G_1 has an Eulerian closed trail*

Eulerian Graphs

Example

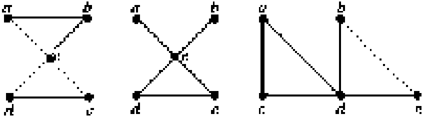


Figure: G_1 G_2 G_3

- *The graph G_1 has an Eulerian closed trail (e.g., a, e, c, d, e, b, a).*
- *Neither G_2 nor G_3 has an Eulerian closed trail.*
- *Note that G_3 has an Eulerian (not closed) trail (e.g., a, c, d, e, b, d, a, b), but there is no Euler trail in G_2.*

Eulerian Graphs

Theorem

The following statements are equivalent for a connected graph G :

(i) G *is Eulerian.*

(ii) *Every point of G has even degree.*

(iii) *The set of lines of G can be partitioned into cycles.*

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Theorem

The following statements are equivalent for a connected graph G :

1. (i) G *is Eulerian.*
2. (ii) *Every point of G has even degree.*
3. (iii) *The set of lines of G can be partitioned into cycles.*

Proof.

- $(i) \Rightarrow (ii)$ Let T be an Eulerian trail in G.

  Each occurrence of a given point in T contributes 2 to the degree of that point, and since each line of G appears exactly once in T, every point must have even degree.

□

Eulerian Graphs

Proof.

- $(ii) \Rightarrow (iii)$ Since G is connected and nontrivial, every point has degree at least 2, so G contains a cycle Z.

  The removal of the lines of Z results in a spanning subgraph G_1 in which every point still has even degree.

  If G_1 has no lines, then (iii) already holds; otherwise, a repetition of the argument applied to G_1 results in a graph G_2 in which again all points are even, etc.

  When a totally disconnected graph G_n is obtained, we have a partition of the lines of G into n cycles.

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- $(iii) \Rightarrow (i)$ is an exercise.

□

Eulerian Graphs

Example

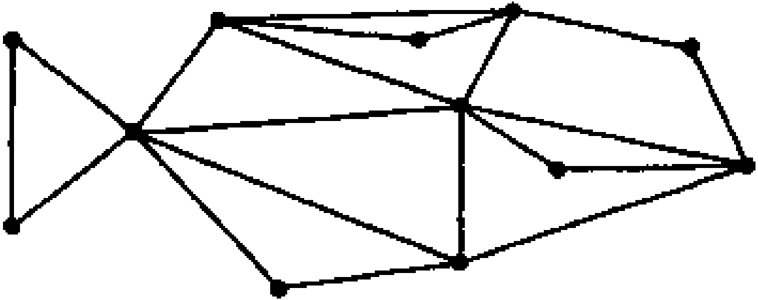


Figure: *An Eulerian Graph*

Hamilton Paths and Circuits

Definition

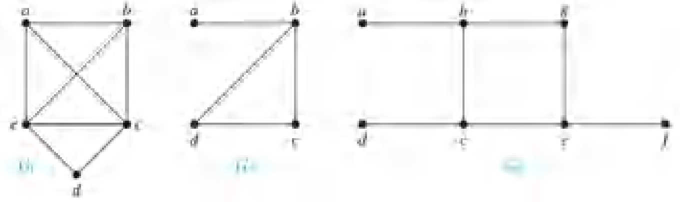
A simple path in a graph G that passes through every vertex exactly once is called a Hamilton[a] path.

A simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

[a]Sir William Hamilton suggested the class of graphs which bears his name when he asked for the construction of a cycle containing every vertex of a dodecahedron.

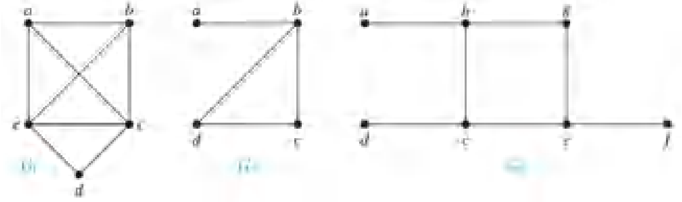
Hamiltonian Graphs

Example



Hamiltonian Graphs

Example



- G_1 *has a Hamilton circuit:* a, b, c, d, e, a.
- G_2 *does not have a Hamilton circuit, but does have a Hamilton path:* a, b, c, d.
- G_3 *does not have a Hamilton circuit, or a Hamilton path.*

Necessary Conditions for Hamiltonian Circuits

- Unlike for an Eulerian circuit, no simple necessary and sufficient conditions are known for the existence of a Hamiton circuit.
- However, there are some useful necessary conditions.

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Theorem (Dirac's Theorem)

If G is a simple graph with $n \geq 3$ vertices s/t the degree of every vertex in G is $\geq \frac{n}{2}$, then G has a Hamilton circuit.

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Theorem (Ore's Theorem)

If G is a simple graph with $n \geq 3$ vertices s/t $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices, then G has a Hamilton circuit.

Applications of Hamiltonian Paths and Circuits

- The famous travelling salesperson problem (TSP) asks for the shortest route a travelling salesperson should take to visit a set of cities. This problem reduces to finding a Hamiltonian circuit s/t the total sum of the weights of its edges is as small as possible.

Applications of Hamiltonian Paths and Circuits

- The famous travelling salesperson problem (TSP) asks for the shortest route a travelling salesperson should take to visit a set of cities. This problem reduces to finding a Hamiltonian circuit s/t the total sum of the weights of its edges is as small as possible.
- Lot of applications of Eulerian and Hamiltonian graph are there in the area of puzzles and games.

The Traveling Salesman Problem (TSP)

- Consider the travelling salesman who wants to visit a number of cities once and then return home.

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   Whether a given graph has a Hamiltonian cycle.
2. *TSP is to find a Hamiltonian cycle with* ***minimum total edge weight*** *in a weighted complete graph. – combinatorial optimization*

Outline

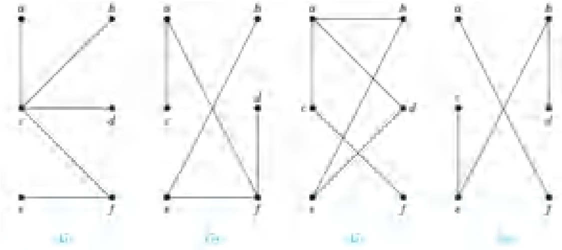
Definition

Definition

- *A graph is acyclic if it has no cycles/circuits.*
- *A tree is a connected acyclic graph.*
- *Any graph without cycles is a forest, thus the components of a forest are trees.*

Example of Trees

Example



Example of Trees

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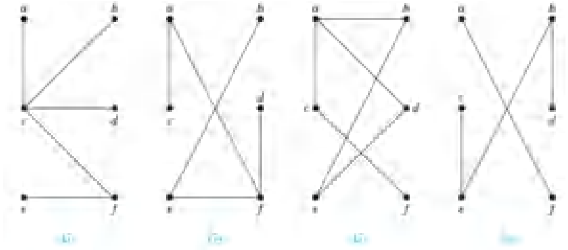
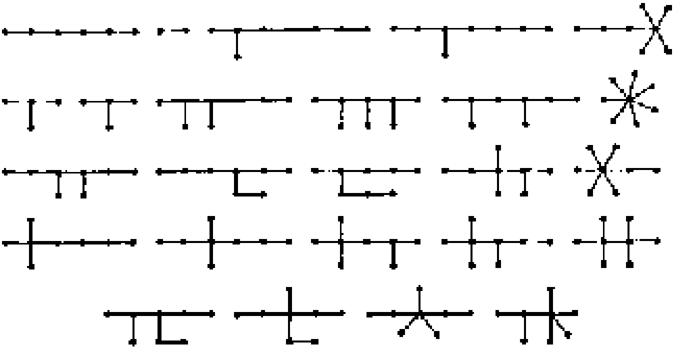


- G_1 *and* G_2 *are trees.*
- G_3 *is not a tree because* e, b, a, d, e *is a circuit in this graph.*
- G_4 *is not a tree because it is not connected.*

Example of Trees

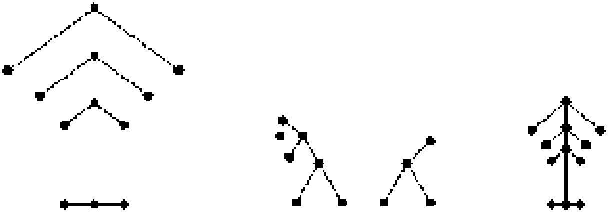
Example



- *There are 23* ***different*** *trees with 8 points*

Example of a Forest

Example



- *This is one graph with three connected components*

Example of a Forest

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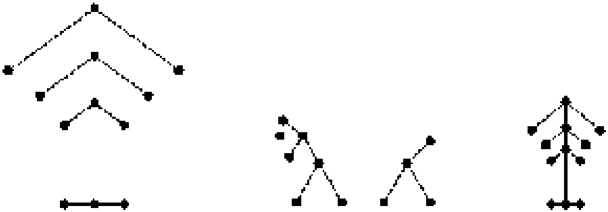

- *This is one graph with three connected components– forest*

Characterization of Trees

Theorem

The following statements are equivalent for a graph G:

1. G *is a tree.*
2. *Every two points of* G *are joined by a unique path.*
3. G *is connected and* $p = q + 1$.
4. G *is acyclic and* $p = q + 1$.
5. G *is acyclic and if any 2 non-adjacent points of* G *are joined by a line* x, *then* $G + x$ *has exactly one cycle.*
6. G *is connected, is not* K_p *for* $p \geq 3$, *and if any 2 non-adjacent points of G are joined by a line* x, *then* $G + x$ *has exactly one cycle.*
7. G *is not* $K_3 \cup K_1$ *or* $K_3 \cup K_2$, $p = q + 1$, *and if any 2 non-adjacent points of* G *are joined by a line* x, *then* $G + x$ *has exactly one cycle.*

Characterization of Trees

Proof.

$(2) \Rightarrow (3)$

- Clearly G is connected.
- We have to prove that $p = q + 1$ (for that we will use induction).
- It is obvious for connected graphs of 1 or 2 points.
- Assume it is true for graphs with fewer than p points.
- If G has p points, the removal of any line of G disconnects G, because of the uniqueness of paths, and in fact this new graph will have exactly two components.
- By the induction hypothesis each component has one more point than line.
- Thus the total number of lines in G must be $p - 1$.

□

Characterization of Trees

Proof.

$(3) \Rightarrow (4)$

- Assume that G has a cycle of length n.
- Then there are n points and n lines on the cycle and for each of the $p - n$ points not on the cycle, there is an incident line on a geodesic to a point of the cycle.
- Each such line is different, so $q \geq p$, which is a contradiction.

□

Characterization of Trees

Proof.

$(4) \Rightarrow (5)$

- Since G is acyclic, each component of G is a tree.
- If there are k components, then, since each one has 1 more point than line, $p = q + k$, so $k = 1$ and G is connected.
- Thus G is a tree and there is exactly one path connecting any two points of G.
- If we add a line uv to G, that line, together with the unique path in G joining u & v, forms a cycle.
- The cycle is unique because the path is unique.

□

Characterization of Trees

Proof.

$(6) \Rightarrow (7)$

- We prove that every two points of G are joined by a unique path and thus, $p = q + 1$.
- Certainly every 2 points of G are joined by some path.
- If 2 points of G are joined by 2 paths, then G has a cycle.
- This cycle cannot have 4 or more points because, if it did, then we could produce more than one cycle in $G + x$ by taking x joining 2 non-adjacent points on the cycle.
- So the cycle is K_3, which must be a proper subgraph of G since by hypothesis G is not complete with $p \geq 3$.
- Since G is connected, we may assume there is another point in G which is joined to a point of this K_3.
- Then it is clear that if any line can be added to G, then one may be added so as to form at least two cycles in $G + x$.
- If no more lines may be added, so that the second condition on G is trivially satisfied, then G is K_p with $p \geq 3$ – a contradiction.

□

Characterization of Trees

Proof.

$(7) \Rightarrow (1)$

- If G has a cycle, that cycle must be a triangle which is a component of G.
- This component has 3 points and 3 lines.
- All other components of G must be trees and, in order to make $p = q + 1$, there can be only one other component.
- If this tree contains a path of length 2, it will be possible to add a line x to G and obtain two cycles in $G + x$. Thus this tree must be either K_1 or K_2.
- So G must be $K_3 \cup K_1$ or $K_3 \cup K_2$, which are the graphs which have been excluded. Thus G is acyclic.
- But if G is acyclic and $p = q + 1$, then G is connected. So G is a tree.

□

Characterization of Trees

Corollary

Every nontrivial tree has at least two endpoints.

Characterization of Trees

Corollary

Every nontrivial tree has at least two endpoints.

A nontrivial tree has $\sum d_i = 2q = 2(p-1)$, there are at least two points with degree less than 2.

Rooted Trees

Definition

A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

An unrooted tree is converted into different rooted trees when different vertices are chosen as the root.

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Terminology for Rooted Trees

Definition

- *If v is a vertex of a rooted tree other than the root, the parent of v is the $!$ vertex u s/t there is a directed edge from u to v. When u is a parent of v, v is called a child of u. Vertices with the same parent are called siblings.*

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- *The ancestors of a vertex are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root. The descendants of a vertex v are those vertices that have v as an ancestor.*

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- *A vertex of a rooted tree with no children is called a leaf. Vertices that have children are called internal vertices.*

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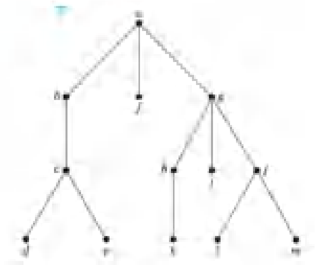
- *A vertex of a rooted tree with no children is called a leaf. Vertices that have children are called internal vertices.*

  If a is a vertex in a tree, the subtree with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

Example of Rooted Trees



- The parent of c is b. The children of g are $h, i, \ \& \ j$. The siblings of h are $i \ \& \ j$. The ancestors of e are $c, b, \ \& \ a$.

  The descendants of b are $c, d, \ \& \ e$.

- The internal vertices are $a, b, c, g, h, \ \& \ j$.

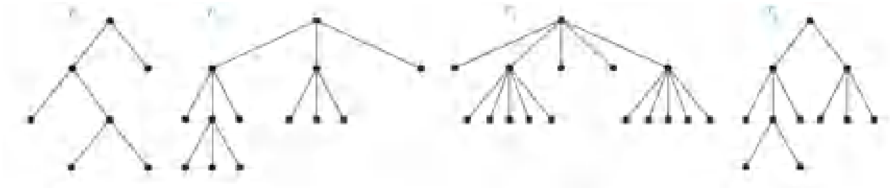
  The leaves are $d, e, f, i, k, l, \ \& \ m$.

m-ary Rooted Trees

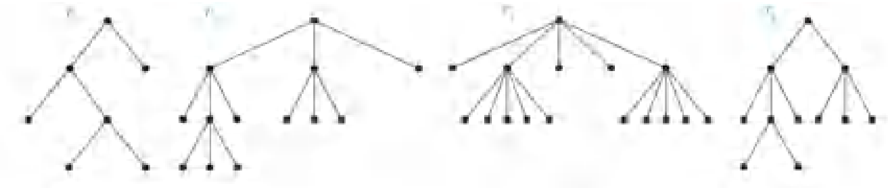
Definition

- *A rooted tree is called an m-ary tree if every internal vertex has no more than m children.*
- *The tree is called a full m-ary tree if every internal vertex has exactly m children.*
- *An m-ary tree with $m = 2$ is called a binary tree.*

Example of m-ary Rooted Trees



Example of m-ary Rooted Trees



- T_1 is a full binary tree because each of its internal vertices has 2 children.
- T_2 is a full 3-ary tree because each of its internal vertices has 3 children.
- In T_3 each internal vertex has 5 children, so T_3 is a full 5-ary tree.
- T_4 is not a full m-ary tree for any m because some of its internal vertices have 2 children and others have 3 children.

Example of m-ary Rooted Trees

Exercise

How many steps do you require to find $x_1 + x_2 + x_3 + \cdots + x_8$*?*

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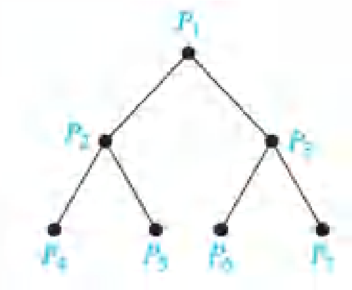

Example of m-ary Rooted Trees

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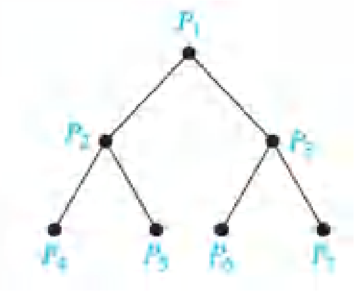

- We require 3 steps using parallel computation

Counting Vertices of m-ary Rooted Trees

Theorem

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices

Counting Vertices of m-ary Rooted Trees

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A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices

Proof.

- Every vertex, except the root, is the child of an internal vertex.
- There are mi vertices in the tree other than the root,

Counting Vertices of m-ary Rooted Trees

Theorem

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices

Proof.

- Every vertex, except the root, is the child of an internal vertex.
- There are mi vertices in the tree other than the root, $\because$ each of the i internal vertices has m children.
- $\therefore$ the tree contains $n = mi + 1$ vertices.

□

Counting Vertices of m-ary Rooted Trees

Theorem

A full m-ary tree with

1. n *vertices has* $i = \frac{n-1}{m}$ *internal vertices and* $\ell = \frac{(m-1)n+1}{m}$ *leaves,*
2. i *internal vertices has* $n = mi + 1$ *vertices and* $\ell = (m-1)i + 1$ *leaves,*
3. ℓ *leaves has* $n = \frac{m\ell-1}{m-1}$ *vertices and* $i = \frac{\ell-1}{m-1}$ *internal vertices.*

Counting Vertices of m-ary Rooted Trees

Proof.

- Let n denote the number of vertices, i the number of internal vertices, and ℓ the number of leaves.
- Then we have $n = mi + 1$ and $n = \ell + i$
- $\Rightarrow \; i = \frac{n-1}{m}$
- $\Rightarrow \ell = n - i$
  $\Rightarrow \ell = n - \frac{n-1}{m}$
  $\Rightarrow \ell = \frac{mn-n+1}{m}$
  $\Rightarrow \ell = \frac{(m-1)n+1}{m}$

□

Level of Vertices and Height of Trees

- When working with trees, we often want to have rooted trees where the subtrees at each vertex contain paths of approximately the same length.

Level of Vertices and Height of Trees

- When working with trees, we often want to have rooted trees where the subtrees at each vertex contain paths of approximately the same length.

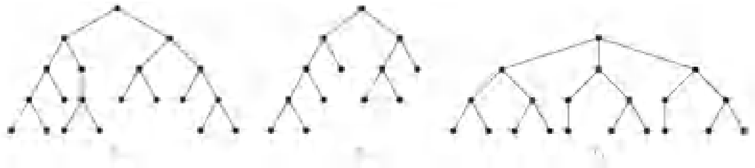
Definition

- *The level of a vertex v in a rooted tree is the length of the ! path from the root to this vertex.*
- *The height of a rooted tree is the maximum of the levels of the vertices.*

Balanced m-ary Trees

Definition

A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h-1$.



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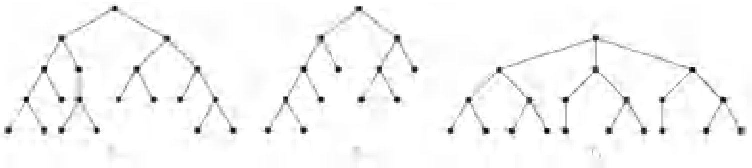

- T_1 is balanced, because all its leaves are at levels 3 and 4.
- T_2 is not balanced, because it has leaves at levels 2, 3, and 4.
- T_3 is balanced, because all its leaves are at level 3.

Bound for the Number of Leaves

Theorem

There are at most m^h leaves in an m-ary tree of height h.

Proof.

Apply mathematical induction on the height, to prove the theorem. □

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The End

Thanks a lot for your attention!