Introduction to Abstract Algebra

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Outline

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Group

Exercise

Solve the following equations:

1. $a + x = b$ & $y + a = b$

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First, we try to solve $a + x = b$

$$\begin{aligned} a + x &= b \\ (-a) + (a + x) &= (-a) + b \end{aligned}$$

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Solve the following equations:

1. $a + x = b$ & $y + a = b$
2. $a.x = b$ & $y.a = b$

Solution

First, we try to solve $a + x = b$

$$\begin{aligned} a + x &= b \\ (-a) + (a + x) &= (-a) + b \\ (-a + a) + x &= -a + b \\ 0 + x &= -a + b \\ x &= -a + b \end{aligned}$$

Binary Operation

Definition

Let X be a non-void set. Then a ***binary operation*** *in X is a function*

$$f : S\,(\subset X \times X) \to X.$$

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Let X be a non-void set. Then a ***binary operation*** *in X is a function*

$$f : S(\subset X \times X) \to X.$$

- Usually, the binary operation f is denoted by '$\circ$' or '$+$' or '$\cdot$' etc.
- If we use '$\circ$' is the binary operation, then $f(x, y)$ is denoted by $x \circ y$
- If $S = X \times X$, then we say that X is **closed** w.r.t. the binary operation

Set & Structure

Definition

A ***set*** *is a well defined collection of objects.*

Definition

An ***algebraic structure*** *is a set together with (a)some binary operation(s).*

Group

Definition

1. *Let G be a non-empty set with a binary operation $\circ$ defined on it. Then $(G, \circ)$ is said to be a* ***groupoid or magma*** *if $\circ$ is closed*
   i.e. if $\circ : G \times G \longrightarrow G$.
2. *A set G with an operation $\circ$ is said to be a* ***semigroup*** *if G is a groupoid and $\circ$ is associative.*
3. *A set G with an operation $\circ$ is said to be a* ***monoid*** *if G is a semigroup and $\exists$ an element $e \in G_m$ s/t $g.e = e.g = g \; \forall \; g \in G$.*
4. *For each $x \in G, \; \exists$ an element $y \in G$ s/t $y \circ x = x \circ y = e$.*
   Usually, y is denoted by x^{-1}.

If G satisfies all the above, it is said to be a ***Group****.*

If $x \circ y = y \circ x \; \forall \; x, y \in G$, G is called abelian or commutative group.

Group

Example

1. $(\mathbb{Z}, +)$
2. $(\mathbb{Q}, +), (\mathbb{Q} \setminus \{0\}, \cdot)$
3. $(\mathbb{R}, +), (\mathbb{C}, +), (\mathbb{R}^*, \cdot), (\mathbb{C}^*, \cdot)$
4. $(\mathbb{Z}_n, +)$
5. $(\mathbb{Z}_p^*, \cdot)$
6. $(\{1, -1\}, \cdot)$
7. $(S_n, \circ)$

Group

Example (S_3)

Let us consider the following important example S_3 under composition of functions.

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Group

Example (S_3)

$\circ$	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

Exercises

Exercise

1. *Give an example of a groupoid which is not a semigroup.*
2. *Give an example of a semigroup which is not a monoid.*
3. *Give an example of a monoid which is not a group.*
4. *Give an example of a semigroup which is not a group.*

Group

Theorem

Let $(G, \circ)$ be a group and e_ℓ be a left identity and for each $x \in G$, x_ℓ^{-1} denote the left inverse of x.

(i) *Then e_ℓ is the $!$ two sided identity in G.*

(ii) *x_ℓ^{-1} is the $!$ two sided inverse of x for each $x \in G$.*

Note:

(a) If e' is any identify whether left or right then $e' = e_\ell$.

(b) If y is any left or right inverse of x then $y = x_\ell^{-1}$.

Some Preliminary Lemmas

Lemma

If $(G, \cdot)$ *[*G*] is a group, then*

1. (i) *The identity element of* G *is* $!$.
2. (ii) *Every* $a \in G$ *has a* $!$ *inverse in* G.
3. (iii) *For every* $a \in G$, $(a^{-1})^{-1} = a$.
4. (iv) *For all* $a, b \in G$, $(a.b)^{-1} = b^{-1}.a^{-1}$

Some Preliminary Lemmas

Lemma

If $(G, \cdot)$ [G] is a group, then

(i) *The identity element of G is $!$.*

(ii) *Every $a \in G$ has a $!$ inverse in G.*

(iii) *For every $a \in G$, $(a^{-1})^{-1} = a$.*

(iv) *For all $a, b \in G$, $(a.b)^{-1} = b^{-1}.a^{-1}$*

Proof.

First, we assume that e & e' are two identities of G.
For every $a \in G$, $e.a = a$. So, $e.e' = e'$, assuming e as an identity element.
Similarly, for every $b \in G$, $b.e' = b$. So, $e.e' = e$, assuming e' as an identity element.
Thus, we have $e' = e.e' = e$, i.e., $e = e'$.

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$$x = e.x = (b.a).x = b.(a.x) = b.(a.y) = (b.a).y = e.y = y$$

□

Some Preliminary Lemmas

Lemma

Let $(G, \circ)$ *be a group and* $c \in G$ *s/t* $c^2 = c$*. Then* $c = e$*, where* e *is the identity element of* G*.*

Some Preliminary Lemmas

Lemma

Let $(G, \circ)$ be a group and $c \in G$ s/t $c^2 = c$. Then $c = e$, where e is the identity element of G.

Proof.

$$\begin{aligned} \because c^2 &= c \\ \therefore c.c &= c \\ \Rightarrow c^{-1}.(c.c) &= c^{-1}.c \\ \Rightarrow (c^{-1}.c).c &= e \\ \Rightarrow e.c &= e \end{aligned}$$

Thus, $c = e$. □

Group

Cancellation Law

Let $(G, \circ)$ be a group. Then for each triplet $x, y, z \in G$

(i) $x \circ y = x \circ z \Rightarrow y = z$ (left cancellation law)

(ii) $y \circ x = z \circ x \Rightarrow y = z$ (right cancellation law)

Subgroup

Definition

A subset H of a group G is said to be a subgroup of G if H itself forms a group under the restricted binary operation in G.

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Lemma

A non-empty subset H of the group G is a subgroup of G iff

(i) $a, b \in H \Rightarrow a.b \in H$;

(ii) $a \in H \Rightarrow a^{-1} \in H$.

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If $(\phi \neq)H \subset G$ *&* $\#H < \infty$ *and* H *is closed under multiplication, then* $H \leq G$.

Subgroup

Lemma

If $(\phi \neq)H \subset G$ & $\#H < \infty$ and H is closed under multiplication, then $H \leq G$.

Proof.

- We need to show that for any $a \in H$, $a^{-1} \in H$
- Suppose $a \in H$, thus, $a^2 = a.a \in H, a^3 = a^2.a \in H, \dots, a^m \in H$. [$\because H$ is closed]
- Thus, the infinite collection of elements $a, a^2, \dots, a^m, \dots$ must all $\in H$ which is a finite subset of G
- $\because H < \infty$, there will be some $r, s \in \mathbb{N}$, $a^r = a^s$. By cancellation law in G, $a^{r-s} = e$, assuming $r > s$.
- $\because (r - s - 1) \geq 0$, $a^{r-s-1} \in H$ and $a^{-1} = a^{r-s-1} \in H$.

□

Note: The lemma may not be true if H is not finite.

Subgroup

Lemma

If $(\phi \neq)H \subset G$ & $\#H < \infty$ and H is closed under multiplication, then $H \leq G$.

Proof.

- We need to show that for any $a \in H$, $a^{-1} \in H$
- Suppose $a \in H$, thus, $a^2 = a.a \in H, a^3 = a^2.a \in H, \ldots, a^m \in H$. [$\because H$ is closed]
- Thus, the infinite collection of elements $a, a^2, \ldots, a^m, \ldots$ must all $\in H$ which is a finite subset of G
- $\because H < \infty$, there will be some $r, s \in \mathbb{N}$, $a^r = a^s$. By cancellation law in G, $a^{r-s} = e$, assuming $r > s$.
- $\because (r - s - 1) \geq 0$, $a^{r-s-1} \in H$ and $a^{-1} = a^{r-s-1} \in H$.

□

Note: The lemma may not be true if H is not finite. $(\mathbb{N}, +)$ and $(\mathbb{Z}^*, \cdot)$

Subgroup

Example

1. $(\mathbb{Z}, +) \leq (\mathbb{R}, +)$
2. $(\mathbb{Q}^*, \cdot) \leq (\mathbb{R}^*, \cdot)$
3. *Let* $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*, where* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc \neq 0$*.* G *is a group under matrix multiplication.*

   $H = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$*, and* $b \in \mathbb{R}$*. Then* $H \leq G$*.*

Subgroup

Proposition

Let $(G, \cdot)$ be a group and T be a non-void subset of G. Then the following are equivalent:

1. $T \leq G$
2. *For each* $x, y \in T$, $x \cdot y$ & $x^{-1} \in T$
3. *For each* $x, y \in T$, $x \cdot y^{-1} \in T$

Subgroup

Definition

Let G be a group and $S, T \subset G$. We then define

$$S \cdot T = \begin{cases} z \in G \mid z = x.y & for\ x \in S,\ \&\ y \in T \\ \phi, & if\ either\ S\ or\ T = \phi \end{cases}$$

$$S^{-1} = \begin{cases} z \in G, & z^{-1} \in S \\ \phi, & if\ S = \phi \end{cases}$$

Subgroup

Proposition

Let G *be a group and* T *be a non-void subset of* G*. Then the following are equivalent:*

1. $T \leq G$
2. $T \cdot T \subset T \;\&\; T^{-1} \subset T$
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Let G be a group and T be a non-void subset of G. Then the following are equivalent:

1. $T \leq G$
2. $T \cdot T \subset T$ & $T^{-1} \subset T$
3. $T \cdot T^{-1} \subset T$

Exercise

Let G be a group and H & $K \leq G$. Then $H \cdot K$ is a subgroup of G iff $H \cdot K = K \cdot H$.

Exercise

Let $\{T_\alpha, \alpha \in \lambda\}$ be a collection of subgroups of G. Then $\bigcap\{T_\alpha, \alpha \in \lambda\}$ is also a subgroup of G.

Subgroup

Solution

- *First, we assume that* $H.K = K.H$ *and we have to prove that* $H.K \leq G$*.*
- *Let* $u, v \in H.K$*. Then* $u = h_1.k_1 \ \& \ v = h_2.k_2$

$$u.v = (h_1k_1).(h_2k_2) = h_1(k_1.h_2)k_2$$

Now, $k_1h_2 \in KH = HK$ *and so* $\exists\ h', k'$ *s/t* $k_1h_2 = h'k'$*,* $h' \in H \ \& \ k' \in K$*.*

$$h_1(k_1.h_2)k_2 = h_1(h'k')k_2 = (h_1h').(k'k_2) = h_3k_3 \in HK,$$

$\because h_3 = h_1h' \in H$ *and* $k_3 = k'k_2 \in K$*.*

- $u^{-1} = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH = HK$
  $\Rightarrow \exists\ h_4 \ \& \ k_4 \ni k_1^{-1}h_1^{-1} = h_4k_4 \in HK$.

So, HK *is a subgroup of* G*.*

Subgroup

Solution

- *First, we assume that* $H.K = K.H$ *and we have to prove that* $H.K \leq G$.
- *Let* $u, v \in H.K$. *Then* $u = h_1.k_1 \ \& \ v = h_2.k_2$

$$u.v = (h_1k_1).(h_2k_2) = h_1(k_1.h_2)k_2$$

Now, $k_1h_2 \in KH = HK$ *and so* $\exists\, h', k'$ *s/t* $k_1h_2 = h'k'$, $h' \in H \ \& \ k' \in K$.

$$h_1(k_1.h_2)k_2 = h_1(h'k')k_2 = (h_1h').(k'k_2) = h_3k_3 \in HK,$$

$\because h_3 = h_1h' \in H$ *and* $k_3 = k'k_2 \in K$.

- $u^{-1} = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH = HK$
  $\Rightarrow \exists\, h_4 \ \& \ k_4 \ni k_1^{-1}h_1^{-1} = h_4k_4 \in HK$.

So, HK *is a subgroup of* G. *Converse part is an exercise.*

Subgroup generated by a subset

Let G be a group and S be a subset of G. Then there is a smallest[1] subgroup T of G containing S. Then T is said to be generated by S and is denoted by $\langle S \rangle$.

Theorem

Let G be a group and S be a non-void subset of G. Then $\langle S \rangle$ consists of all finite product of the form

$$x_1.x_2.\dots.x_n, \; for \; n \in \mathbb{N} \; \& \; x_i \in S \cup S^{-1}.$$

[1] T is the smallest in the following sense:
if H is a subgroup and $S \subset H$ then $T \subset H$

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$$x_1.x_2.\ldots.x_n,\ for\ n \in \mathbb{N}\ \&\ x_i \in S \cup S^{-1}.$$

Theorem

If G is an abelian group and $(\phi \neq)S \subset G$, then $\langle S \rangle$ consists of all elements of the form $x_1^{r_1}.x_2^{r_2}.\ldots.x_k^{r_k}$, $x_i \neq x_j, r_i \in \mathbb{Z}$.

[1] T is the smallest in the following sense: if H is a subgroup and $S \subset H$ then $T \subset H$

Cyclic Group

Theorem

Let G be a group and $a \in G$. Then $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G and is the smallest subgroup of G that contains a.

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Theorem

Let G be a group and $a \in G$. Then $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G and is the smallest subgroup of G that contains a.

Definition

1. *Let G be a group and $a \in G$. Then the smallest subgroup $H = \{a^n \mid n \in \mathbb{Z}\}$ of G which contains a is called the cyclic subgroup of G generated by a.*
2. *An element $a \in G$ generates G and is a generator for G if $\langle a \rangle = G$.*
3. *A group G is cyclic if there is some element $a \in G$ that generates G.*

Subgroup

Notation:

- a^n under multiplication $a^n = \overbrace{a.a.\cdots.a}^{n-times}$
- a^n under addition $a^n = n.a = \underbrace{a + a + \cdots + a}_{n-times}$
- $a.b^{-1}$ under addition

Subgroup

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- a^n under multiplication $a^n = \overbrace{a.a.\cdots .a}^{n-times}$
- a^n under addition $a^n = n.a = \underbrace{a + a + \cdots + a}_{n-times}$
- $a.b^{-1}$ under addition $a - b$

Group

Definition

1. *A group G is finite if $|G|$ or $\# G$ is finite. The number of elements in a finite group is called its order.*
2. *A group G is cyclic if $\exists\ \alpha \in G$ s/t for each $\beta \in G\ \exists$ integer i with $\beta = \alpha^i$. Such an element α is called a generator of G.*
3. *Let $\alpha \in G$. The order of α is defined to be the least positive integer t s/t $\alpha^t = e$, provided that such an integer exists. If such a t does not exist, then the order of α is defined to be ∞.*

Subgroup

Example

1. *Consider the multiplicative group* $\mathbb{Z}_{19}^{*} = \{1, 2, \cdots, 18\}$ *of order 18.*

Subgroup	Generators	Order
$(\{1\}, \cdot)$	1	1
$(\{1, 18\}, \cdot)$	18	2
$(\{1, 7, 11\}, \cdot)$	7, 11	3
$(\{1, 7, 8, 11, 12, 18\}, \cdot)$	8, 12	6
$(\{1, 4, 5, 6, 7, 9, 11, 16, 17\}, \cdot)$	4, 5, 6, 9, 16, 17	9
$(\mathbb{Z}_{19}^{*}, \cdot)$	2, 3, 10, 13, 14, 15	18

2. *Consider the multiplicative group* $(\mathbb{Z}_{26}^{*}, \cdot)$

Coset

Definition

*Let G be a group and $H \leq G$. For $a, b \in G$, we say that a is **congruent to** $b \mod H$, i.e., $a \equiv b \mod H$ if $a.b^{-1} \in H$.*

Lemma

The relation $a \equiv b \mod H$ is an equivalence relation.

Definition

If $H \leq G, a \in G$, then

$$Ha = \{ha \mid h \in H\} \ [aH = \{ah \mid h \in H\}].$$

Ha is called a right [left] coset of H in G.

Coset

Lemma

If $H \leq G$, then

$$Ha = \{x \in G \mid a \equiv x \mod H\}$$

Coset

Lemma

If $H \leq G$, then

$$Ha = \{x \in G \mid a \equiv x \mod H\}$$

Proof.

Let $[a] = \{x \in G \mid a \equiv x \mod H\}$. First, we prove that $Ha \subset [a]$.
If $h \in H$, $ha \in H$. Now we see $a(ha)^{-1} = a(a^{-1}h^{-1}) = h^{-1} \in H$, $\because H \leq G$.
By definition of congruence, $ha \in [a]$ for every $h \in H$ and so $Ha \subset [a]$.

Next we assume that $x \in [a]$. Thus $ax^{-1} \in H$, so $(ax^{-1})^{-1} = xa^{-1} \in H$, i.e., $xa^{-1} = h$ for some $h \in H$.
$(xa^{-1})a = ha \Rightarrow x = ha$.
Thus, $[a] \subset Ha$.

Thus, we have $[a] = Ha$.

Coset

Lemma

If $H \leq G$, then

$$Ha = \{x \in G \mid a \equiv x \mod H\}$$

Proof.

Let $[a] = \{x \in G \mid a \equiv x \mod H\}$. First, we prove that $Ha \subset [a]$.
If $h \in H$, $ha \in H$. Now we see $a(ha)^{-1} = a(a^{-1}h^{-1}) = h^{-1} \in H, \because H \leq G$.
By definition of congruence, $ha \in [a]$ for every $h \in H$ and so $Ha \subset [a]$.

Next we assume that $x \in [a]$. Thus $ax^{-1} \in H$, so $(ax^{-1})^{-1} = xa^{-1} \in H$, i.e., $xa^{-1} = h$ for some $h \in H$.
$(xa^{-1})a = ha \Rightarrow x = ha$.
Thus, $[a] \subset Ha$.

Thus, we have $[a] = Ha$. □

Thus, any 2 right cosets of H in G are either identical or have no element in common.

Coset

Exercise

Prove that there exists a bijection $f : aH \to Hb$ *and hence there exists a bijection from* $aH \leftrightarrow bH$*, for any* $a, b \in G$.

Coset

Exercise

Prove that there exists a bijection $f : aH \to Hb$ and hence there exists a bijection from $aH \leftrightarrow bH$, for any $a, b \in G$.

Solution

Hint:

- $f : aH \to Hb$ given by $u \mapsto a^{-1}ub$
- *Prove that f is injective as well as onto.*

Coset

Exercise

Prove that there exists a bijection $f : aH \to Hb$ and hence there exists a bijection from $aH \leftrightarrow bH$, for any $a, b \in G$.

Solution

Hint:

- $f : aH \to Hb$ given by $u \mapsto a^{-1}ub$
- *Prove that f is injective as well as onto.*
- *By taking $b = e$, there is a bijection $f_a : aH \to H$.*
- *So, there is a bijection $f_b : bH \to H$.*
- *Then $f_b^{-1} \circ f_a : aH \to bH$ is a bijection.*

Coset

Proposition

Let G be a group and $H \leq G$ & $a, b \in G$. The following are equivalent:

i) $a.H = b.H$

ii) $a^{-1}b \in H$ [or $b^{-1}a \in H$]

iii) $a \in b.H$ [or $b \in a.H$]

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iii) $a \in b.H$ [or $b \in a.H$]

Proof.

Hint:

- $(i) \Rightarrow (ii)$
  $b \in bH = aH$. So, $\exists\, h \in H \ni b = ah$
- $(ii) \Rightarrow (iii)$
  $b^{-1}a \in H \Rightarrow \exists\, h \in H \ni b^{-1}a = h$
- $(iii) \Rightarrow (i)$
  $\because a \in bH \therefore a = bh_0$, for some $h_0 \in H$. Now, PT $aH \subset bH$ & $bH \subset aH$

□

Coset

Theorem

Let G be a group and $H \leq G$. For each $a \in G$,

- i) $a \in aH$
- ii) *For any pair $a, b \in G$, either $aH = bH$ or $aH \cap bH = \phi$*
- iii) $\bigcup\{aH \ni a \in G\} = G$
- iv) $\{aH \ni a \in G\}$ *is a partition of G.*

Coset

Theorem

Lagrange's Theorem: *If* G *is a finite group &* $H \leq G$ *, then*

$$\#H \mid \#G \; [or \;\; \circ(H) \mid \circ(G)]$$

Hence, if $a \in G$*, the order of* a *divides* $\#G$*.*

Coset

Theorem

Lagrange's Theorem: *If G is a finite group & $H \leq G$, then*

$$\#H \mid \#G \; [or \;\; \circ(H) \mid \circ(G)]$$

Hence, if $a \in G$, the order of a divides $\#G$.

Proof.

- Let $x_1H, x_2H, \ldots$ be the set of distinct left cosets of H in G
- $\bigcup_{i=1}^{k} x_iH = G$ and $x_iH \cap x_jH = \phi$ for $i \neq j$
- $\because |x_iH| = |H| = m$ (say)
- $\therefore |G| = \sum_{i=1}^{k} |x_iH| = \sum_{i=1}^{k} m = mk = n$ (say)

$$\#H \mid \#G$$

□

Subgroup

Corollary

1. *Let* $(G, \cdot)$ *be a finite group of order* p*, where* p *is a prime. Then* G *is cyclic and hence abelian.*
2. *Let* $(G, \cdot)$ *be a finite group and* $x \in G$ *be an arbitrary element. Then* $order\ of\ x$ *is a divisor of* $order\ of\ G$*.*
3. *Let* p *be a prime number and* $\gcd(a, p) = 1$*, where* $a \in \mathbb{N}$*. Then* $a^{p-1} \equiv 1 \mod p$*.*
4. *Let* p *be prime. Then* $(p-1)! \equiv -1 \mod p$*.*

Subgroup

Proof.

- We know that $(\mathbb{Z}_p^*, \cdot)$ is a group of order $p - 1$.
- Show that the only element of order 2 in $\mathbb{Z}_p^*$ is

Subgroup

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- We know that $(\mathbb{Z}_p^*, \cdot)$ is a group of order $p-1$.
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Subgroup

Proof.

- We know that $(\mathbb{Z}_p^*, \cdot)$ is a group of order $p-1$.
- Show that the only element of order 2 in $\mathbb{Z}_p^*$ is $p-1$
- Consider

$$2.3\ldots(p-2) = 1,$$

$\because$ none of these elements are self inverse.

- Thus, we have

$$\begin{aligned}(p-2)! &\equiv 1 \mod p\\ (p-1)! &\equiv (p-1) \mod p\\ &\equiv -1 \mod p\end{aligned}$$

Cyclic Group

Theorem

Every subgroup H of a cyclic group G is also cyclic.

In fact, if G is a cyclic group of order n, then for each positive divisor d of n, G contains exactly one subgroup of order d.

- Let $\langle a \rangle = G$.
- If H is $\{e\}$, then there is nothing to prove. So, we assume $H \neq \{e\}$.
- Then $\exists\ u \in H \ni u \neq e$
- We have now 2 cases:

Case-1: G is infinite cyclic group

- $\exists\ n_0 \ni u = a^{n_0}$.
- $\because u \in H \Rightarrow u^{-1} \in H$ as $H \leq G$
- Let $T = \{n \in \mathbb{N}\ :\ n > 0,\ a^n \in H\}$
- $T \neq \phi$

Cyclic Group

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- Let $T = \{n \in \mathbb{N}\ :\ n > 0,\ a^n \in H\}$
- $T \neq \phi$ as $n_0\ or\ -n_0 \in T$

Cyclic Group

Case-1: G is infinite cyclic group

- $\because \mathbb{N}$ is well-ordered, $\therefore T$ has a least element, say k_0.
- Then $a^{k_0} \in H$ and $1 \leq n < k_0, a^n \notin H$
- Again, let M be a cyclic group generated by a^{k_0}
- Then $\because a^{k_0} \in H$ and H is a subgroup, $M \subset H$
- Now, let $v \in H$. Then $v = a^m$ for $m \in \mathbb{Z}$

  $m = qk_0 + r$, where $0 \leq r < k_0$
- Now, $a^m \in H$ and $a^{qk_0} = (a^{k_0})^q \in H$
  So, $a^{m-qk_0} \in H \Rightarrow a^r \in H$
- By minimal property of k_o we must have $r = 0$. So $m = qk_0$
- Then, $a^m = (a^{k_0})^q \in M$. Then $H \subset M \Rightarrow M = H$.

Thus, H is a cyclic subgroup generated by a^{k_0}.

Cyclic Group

Case-2: G is finite cyclic group of order m

- Then $G = \{e, a, a^2, \dots a^{m-1}\}$.
- Let $T = \{r \in \mathbb{N} \ : \ a^r \in H, \ 1 \leq r \leq m-1\}$
- Then $T \neq \phi \ \because H \neq \phi$.
- Let k_0 be the minimum value of r, s/t $a^r \in H$.
- $a^{k_0} \in H$.
- Then by above H is cyclic subgroup generated by a^{k_0}.

Cyclic Group

Example

1. $(\mathbb{Z}, +)$ *and* $(\mathbb{Z}_n, +)$ *are cyclic groups*
2. $(\mathbb{Z} \times \mathbb{Z}, +)$ *is not cyclic group. However, it is finitely generated.*

Cyclic Group

Example

1. $(\mathbb{Z}, +)$ *and* $(\mathbb{Z}_n, +)$ *are cyclic groups*
2. $(\mathbb{Z} \times \mathbb{Z}, +)$ *is not cyclic group. However, it is finitely generated.* $S = \{(1, 0), (0, 1)\}$ *generates* $\mathbb{Z} \times \mathbb{Z}$
3. $(\mathbb{Q}, +)$ & $(Q^*, \cdot)$ *are not finitely generated.*

Homomorphism

Definition

Let $(G_1, \cdot)$ *and* $(G_2, \cdot)$ *be groups and* $f : G_1 \to G_2$ *be a function. Then*

1. f *is said to be a homomorphism iff for each* $a, b \in G_1$,
$$f(a.b) = f(a).f(b).$$
2. *A homomorphism* f *is said to be monomorphism (epimorphism) iff* f *is injective (surjective).*
3. *A homomorphism* f *is said to be isomorphism iff* f *is both monomorphism and an epimorphism.*

Homomorphism

Proposition

Let G_1, G_2, G_3 *be groups and* $f : G_1 \to G_2$ & $g : G_2 \to G_3$ *be homomorphisms.*

Then $g \circ f : G_1 \to G_3$ *is also a homomorphism.*

Further, $g \circ f$ *is a monomorphism (epimorphism) if* g & f *are both injective (surjrctive).*

Thus, in particular if f & g *are isomorphisms, so is* $g \circ f$.

Also, if f *is isomorphism from* $G_1 \to G_2$*, then* $f^{-1} : G_2 \to G_1$ *is also an isomorphism.*

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Note: Let C be collections of groups. Define $G_1 \sim G_2$ $(G_i \in C)$ iff $\exists$ an isomorphism $f : G_1 \to G_2$. Verify that $\sim$ is an equivalence relation.

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Note: Let C be collections of groups. Define $G_1 \sim G_2$ $(G_i \in C)$ iff $\exists$ an isomorphism $f : G_1 \to G_2$. Verify that $\sim$ is an equivalence relation.

Two isomorphic groups are absolutely indistinguishable. The main problem of group theory is to decide whether to given groups are isomorphic or not

Homomorphism

Exercise

Let P be the set of all polynomials with integer coefficient. Then $(P, +)$ is a abelian group. Show that $(P, +)$ is isomorphic to $(\mathbb{Q}^*, \cdot)$. [$(P, +) \cong (\mathbb{Q}^*, \cdot)$]

Homomorphism

Exercise

Let P be the set of all polynomials with integer coefficient. Then $(P,+)$ is a abelian group. Show that $(P,+)$ is isomorphic to $(\mathbb{Q}^*,\cdot)$. [$(P,+)\cong(\mathbb{Q}^*,\cdot)$]

Solution

- *Let $\{p_n\}_{n=0}^{\infty}$ be the set of all primes enumerated as*

$$p_0 < p_1 < p_2 < \cdots$$

- *Now, we define $f:(P,+)\to(\mathbb{Q}^*,\cdot)$ as follows:*
  for $p(x)\in P$, with $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

$$f(p(x)) = p_0^{a_0}.p_1^{a_1}.p_2^{a_2}\ldots p_n^{a_n}$$

- *Show that f is an isomorphism.*

Detailed Study of Cyclic Group

Theorem

Let $(G, \cdot)$ be a cyclic group[a]. Then

(i) $(G, \cdot) \cong (\mathbb{Z}, +)$ *iff* G *is infinite*

(ii) $(G, \cdot) \cong (\mathbb{Z}_n, +)$ *iff* G *is finite and* $|G| = n$.

[a] This is the complete characterization theorem for cyclic group

Detailed Study of Cyclic Group

Theorem

Let $(G, \cdot)$ be a cyclic group[a]. Then

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[a] This is the complete characterization theorem for cyclic group

Proof.

Let G be a cylic group generated by a. Then $G = \{a^n \ : \ n \in \mathbb{Z}\}$. Then two cases can arise

Case-1: $a^n \neq a^m$ for $n \neq m$

Consider the function $f : (\mathbb{Z}, +) \to (G, \cdot)$ given by $m \mapsto a^m$

Case-2: $\exists\, n, m \in \mathbb{Z} \ni a^n = a^m$

Consider the function $f : (\mathbb{Z}_n, +) \to (G, \cdot)$ given by $\bar{m} \mapsto a^{\bar{m}}$

□

Cyclic Group

Exercise

1. *Let* G *be a group.*
   - (a) *If the order of* $a \in G$ *is* t, *then the order of* a^k *is* $\frac{t}{gcd(t,k)}$.
   - (b) *If* G *is a cyclic group of order* n & $d \mid n$, *then* G *has exactly* $\phi(d)$ *elements of order* d. *In particular,* G *has* $\phi(n)$ *generators.*
2. *Let* G_1, G_2 *be cyclic group of order* m, n *respectively and* $\gcd(m,n) = 1$. *Then* $G_1 \times G_2$ *is a cyclic group of order* mn.

   If $\gcd(m,n) \neq 1$, $G_1 \times G_2$ *is never cyclic.*

Normal Subgroup

Definition

If $H \leq G$, the index of H in G is the number of distinct right (or left) cosets of H in G.

We denote it by $i_G(H)$. In case G is a finite group,

$$i_G(H) = \frac{\circ(G)}{\circ(H)}.$$

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Definition

Let G be a group and H be a subgroup of G. Then H is said to be a normal [or invariant] subgroup of G iff for each $x \in G$, $xH = Hx$. [$H \trianglelefteq G$]

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If G is abelian, then every subgroup is normal.

Normal Subgroup

- If G is non-abelian, it may happen that $aH \neq Ha$ for some $a \in G$.
- Consider the group $(S_3, \circ)$

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$
$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

- Let

$$H = \left\{ \rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\} \& a = \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Quotient Group

Theorem

Let G be a group and H be a normal subgroup of G. Then the G/H of left cosets of H in G is a group under operation of set product.

Proof.

Hint:

- Let xH & $yH \in G/H$. Prove that $(xH)(yH) \in G/H$
- The element $H = eH$ is the identity element of G/H
- Prove that $x^{-1}H$ is the inverse of xH

□

Definition

The G/H is called the quotient group of G by the normal subgroup H.

Quotient Group

Exercise

Let $(\mathbb{Z}, +)$ *be the additive group of integers. Any subgroup of* $\mathbb{Z}$ *is of the form* $n\mathbb{Z}$ *for* $n \in \mathbb{Z}^+$. *Then* $n\mathbb{Z}$ *is a normal subgroup.*

Show that $(\mathbb{Z}/n\mathbb{Z}, +) = (\mathbb{Z}_n, +)$.

Proposition

Let $(G_1, \cdot), (G_2, \cdot)$ *be two groups and* $f : G_1 \to G_2$ *be a homomorphism. Then*

(i) $f(e_1) = e_2$, *where* e_1, e_2 *are the identities of* G_1, G_2 *respectively.*

(ii) *For each* $x \in G_1$, $f(x^{-1}) = (f(x))^{-1}$

(iii) *If* $T \leq G_1$, $f(T) \leq G_2$

First Isomorphism Theorem

Theorem

Let G_1 & G_2 *be two groups and* $f : G_1 \to G_2$ *be a homomorphism.*

Let $K = \{x \in G_1 \ : \ f(x) = e_2\}$ *denote the kernel of* f

Then,

(i) $K \trianglelefteq G_1$

(ii) *The quotient group* G_1/K *is isomorphic to image of* $f = f(G_1)(\subset G_2)$ *under the following map*

$$\tilde{f} : G_1/K \to G_2 \; defined \; by \; \tilde{f}(xK) = f(x)$$

Second Isomorphism Theorem

Theorem

Let $(G, \cdot)$ *be a group and* H & $K \leq G$ *of which* $K \trianglelefteq G$.

Then,

(i) $H.K \leq G$

(ii) $H \cap K \trianglelefteq H$.

(iii) $H.K/K \cong H/H \cap K$

Third Isomorphism Theorem

Theorem

Let $(G, \cdot)$ *be a group and* $H \ \& \ K \trianglelefteq G$ *s/t* $K \subset H$.

Then the quotients groups $G/K, G/H,$ *and* H/K *are defined and* H/K *is a normal subgroup of* G/K *and further*

$$G/H \cong (G/K)/(H/K)$$

Outline

Rings

Definition

A ***ring*** $(R, +, \cdot)$ *is a set* R *with 2 binary operations addition* $+$ *and multiplication* $\cdot$ *defined on* R *s/t the following conditions are satisfied:*

1. (i) $(R, +)$ *is an abelian group*
2. (ii) *multiplication* $\cdot$ *is associative*
3. (iii) *For all* $a, b, c \in R$ *the* ***left distributive law***

$$a.(b + c) = (a.b) + (a.c)$$

and ***right distributive law***

$$(a + b).c = (a.c) + (b.c) \; hold$$

Rings

Definition

1. *If a ring* R *contains the identity element* 1 *w.r.t. to multiplication, i.e.,* $1.a = a.1 = a \; \forall \; a \in R$*, then we shall describe* R *as a ring with unit element or ring with identity.*
2. *If the multiplication* $\cdot$ *is commutative on* R*, i.e.,* $a.b = b.a \; \forall \; a, b \in R$*, then we call* R *is a commutative ring.*
3. *If* R *satisfied both the above conditions, the we say* R *is a commutative ring with identity.*

Rings

Example

1. $R = (\mathbb{Z}, +, \cdot)$ – *the set of integers under the usual rules of addition and multiplication forms a ring.* R *is commutative ring with identity*[a].
2. R *is the set of even integers under the usual rules of addition and multiplication forms a ring.* R *is commutative ring but has no identity element.*
3. *For* $n \geq 1$, *the set* $\mathbb{Z}_n$ *under modular addition and modular multiplication forms a ring.*
   - (a) *For* $n = 6$, *the set* $\mathbb{Z}_6$ *under modular addition and modular multiplication forms a ring.*
   - (b) *For* $n = 7$, *the set* $\mathbb{Z}_7$ *under modular addition and modular multiplication forms a ring.*

[a] Hilbert first introduced the term **ring**

Rings

Example

4 *The set* $\mathbb{Q}$ *of rational numbers under the usual rules of addition and multiplication forms a ring.*

5 *The set* $\mathbb{R}$ *of real numbers under the usual rules of addition and multiplication forms a ring.*

6 *The set* $\mathbb{C}$ *of complex numbers under the usual rules of addition and multiplication forms a ring.*

7 *Let* $M_n(R)$ *be the collection of all* $n \times n$ *matrices having elements of* R*. Then* $M_n(R)$ *forms a non-commutative ring with matrix addition and matrix multiplication*

(a) $M_n(\mathbb{Z}), M_n(\mathbb{Q}), M_n(\mathbb{R}),\ \&\ M_n(\mathbb{C})$ *form rings under matrix addition and matrix multiplication*

Rings

Example (Ring of Quaternions)

Let Q be the sent of all symbols of the form $\alpha_0 + \alpha_1.i + \alpha_2.j + \alpha_3.k$, where all $\alpha_i \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Let $\alpha, \beta \in Q$ and $\alpha = \alpha_0 + \alpha_1.i + \alpha_2.j + \alpha_3.k$ and $\beta = \beta_0 + \beta_1.i + \beta_2.j + \beta_3.k$.

We define

$\alpha = \beta \iff \alpha_i = \beta_i \;\; for\ i = 0, 1, 2, 3.$

$\alpha + \beta = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1).i + (\alpha_2 + \beta_2).j + (\alpha_3 + \beta_3).k$

$$\alpha.\beta = (\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) + (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2)i + (\alpha_0\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_0 + \alpha_3\beta_1)j + (\alpha_0\beta_3 + \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_0)k$$

Q forms a non-commutative ring under the operations defined above.

Rings

Definition

1. *If* R *is a commutative ring and* $a(\neq 0) \in R$, *then* a *is said to be a zero-divisor, if* $\exists\ b \in R$ *and* $b \neq 0$ *s/t* $a.b = 0$.

Rings

Definition

1. *If R is a commutative ring and $a(\neq 0) \in R$, then a is said to be a zero-divisor, if $\exists\ b \in R$ and $b \neq 0$ s/t $a.b = 0$.*

   For example in $\mathbb{Z}_6$, $2, 3, 4$ are zero-divisors.

2. *A commutative ring is an integral domain if it has no zero-divisors.*

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   For example, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ & $\mathbb{Z}_7$ are integral domains.

3. *A ring is said to be a division ring (or skew field) if its non-zero elements form a group under multiplication.*

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3. *A ring is said to be a division ring (or skew field) if its non-zero elements form a group under multiplication.*

   For example, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and ring of quaternions Q are division rings

Rings & Fields

Definition

The characteristic of an integral domain R is defined as the smallest positive integer m s/t $m.a = 0$ for all $a \in R$.

The characteristic of an integral domain R is defined 0, if we don't have such m.

Definition

A field is a commutative division ring.

*A **field** $(F, +, \cdot)$ satisfies the following conditions:*

(i) $(F, +)$ *is an abelian group*

(ii) $(F \setminus \{0\}, \cdot)$ *is also an abelian group*

(iii) *For all $a, b, c \in F$ the **distributive law***

$$a.(b + c) = (a.b) + (a.c) \;\; hold$$

Rings

Lemma

If R is a ring, then for all $a, b \in R$

- (i) $a.0 = 0.a = 0$
- (ii) $a(-b) = (-a)b = -(ab)$
- (iii) $(-a)(-b) = ab$

If, in addition, R has an identity element 1, then

- (iv) $(-1)a = -a$
- (v) $(-1)(-1) = 1$

Rings & Fields

Lemma

A finite integral domain is a field.

Rings & Fields

Lemma

A finite integral domain is a field.

Proof.

- Let D be a finite integral domain.
- To prove D is a field we must show:
  - $\exists\ 1 \in D$ s/t $a.1 = a\ \forall\ a \in D$
  - For every $a \neq 0 \in D,\ \exists\ b \in D$ s/t $a.b = 1$

Rings & Fields

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  - For every $a \neq 0 \in D,\ \exists\ b \in D$ s/t $a.b = 1$
- Let $x_1, x_2, \ldots x_n$ be all the elements of D, and $a \neq 0 \in D$.
- Consider the elements $x_1 a, x_2 a, \ldots x_n a \in D$.
- **Claim:** they are all distinct!

Rings & Fields

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Proof.

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- To prove D is a field we must show:
  - $\exists\ 1 \in D$ s/t $a.1 = a\ \forall\ a \in D$
  - For every $a \neq 0 \in D,\ \exists\ b \in D$ s/t $a.b = 1$
- Let $x_1, x_2, \dots x_n$ be all the elements of D, and $a \neq 0 \in D$.
- Consider the elements $x_1 a, x_2 a, \dots x_n a \in D$.
- **Claim:** they are all distinct!
- By the pigeonhole principle, $\exists\ i_0$ for which we will have $x_{i_0} a = a$.
- Prove that x_{i_0} is the multiplicative identity, i.e., for any $y \in D,\ y.x_{i_0} = y$

□

Rings & Fields

Corollary

If p is a prime number, then $\mathbb{Z}_p$ is a field.

Rings & Fields

Corollary

If p is a prime number, then $\mathbb{Z}_p$ is a field.

Note: $\mathbb{Z}_n$ never forms a field if n is composite

Exercise

If D is an integral domain and D is of finite characteristic, prove that the characteristic of D is a prime number.

Rings

Example

Let R be a ring and x be an indeterminate. The polynomial ring $R[x]$ is defined to be the set of all formal sums $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$, where $a_i \in R$ are called the coefficients of x^i rsp.

Given two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ & $g(x) = \sum_{i=0}^{m} b_i x^i \in R[x]$

$$f(x) + g(x) = \sum_{i=0}^{n} (a_i + b_i) x^i,$$

where we have implicitly assumed that $m \leq n$ and we set $b_i = 0$, for $i > m$ and

$$f(x).g(x) = \sum_{i=0}^{m+n} \left(\sum_{j=0}^{i} a_{i-j} b_j x^i \right)$$

$R[x]$ becomes a ring, with 0 given as the polynomial with zero coefficients.
If R has identity, $1 \neq 0$ then $R[x]$ has identity, $1 \neq 0, 1$ is the polynomial whose constant coefficient is 1 and other terms are 0.

Rings

Example

Solve $x^2 - 5x + 6 = 0$ *in* Z_{12}.

Rings

Example

Solve $x^2 - 5x + 6 = 0$ *in* Z_{12}.

Solution

$$x^2 - 5x + 6 = (x-2)(x-3) =$$

Rings

Example

Solve $x^2 - 5x + 6 = 0$ *in* Z_{12}.

Solution

$$x^2 - 5x + 6 = (x-2)(x-3) = (x+10)(x+9) = 0$$

$$2.6 = 6.2 = 3.4 = 4.3 = 3.8 = 8.3 = 4.6 = 6.4 = 4.9 = 9.4$$
$$= 6.6 = 6.8 = 8.6 = 6.10 = 10.6 = 8.9 = 9.8 = 0$$

Rings

Example

Solve $x^2 - 5x + 6 = 0$ *in* Z_{12}.

Solution

$$x^2 - 5x + 6 = (x-2)(x-3) = (x+10)(x+9) = 0$$

$$2.6 = 6.2 = 3.4 = 4.3 = 3.8 = 8.3 = 4.6 = 6.4 = 4.9 = 9.4$$
$$= 6.6 = 6.8 = 8.6 = 6.10 = 10.6 = 8.9 = 9.8 = 0$$

Exercise

1. *Find all the solution of the equation* $x^2 + 2x + 4 = 0$ *in* $\mathbb{Z}_6$
2. *Solve the equation* $3x = 2$ *in* $\mathbb{Z}_{23}$

Applications to $ax \equiv b \mod m$

Theorem

Let $m \in \mathbb{N}$ *and* $a \in \mathbb{Z}_m$ *s/t* $\gcd(a, m) = 1$. *For each* $b \in \mathbb{Z}_m$, *the equation* $ax = b$ *has unique solution in* $\mathbb{Z}_m$.

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Let $m \in \mathbb{N}$ and $a, b \in \mathbb{Z}_m$. Let $d = \gcd(a, m)$. The equation $ax = b$ has a solution in $\mathbb{Z}_m$ iff $d \mid b$. When $d \mid b$, the equation has exactly d solutions in $\mathbb{Z}_m$.

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Proof.

- Let $s \in \mathbb{Z}_m$ be a solution of the equation $ax = b$ in $\mathbb{Z}_m$
- $as - b = qm$ in $\mathbb{Z}$

  $b = as - qm$

  $d \mid (as - qm)$
- Thus, a solution s can exist only if $d \mid b$

□

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Proof.

- Suppose $d \mid b$, $\Rightarrow b = b_1 d$
- $\because \gcd(a, m) = d, \therefore a = a_1 d \ \& \ m = m_1 d$
- Then the equation $ax = b$ in $\mathbb{Z}_m$ can be written as $ax - b = qm$ in $\mathbb{Z}$
- $ax - b = qm \Rightarrow d(a_1 x - b_1) = dqm_1$
- Now, $m \mid (ax - b) \iff m_1 \mid (a_1 x - b_1)$
- Thus the solution s of $ax = b$ in $\mathbb{Z}_m$ are precisely the solution of $a_1 x = b_1$ in $\mathbb{Z}_{m_1}$
- Now, $s \in \mathbb{Z}_{m_1}$ is the $!$ solution of $a_1 x = b_1$ in $\mathbb{Z}_{m_1}$
- The numbers $\in \mathbb{Z}_m$ that reduces to $s \mod m_1$

$$s, s + m_1, s + 2m_1, \ldots, s + (d - 1)m_1$$

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Thus, there are exactly d solutions of the equation in $\mathbb{Z}_m$.

□

Ring $(\mathbb{Z}_{26}, +, \cdot)$ in Affine Cipher

- An **affine cipher :**

$$f_{a,b} : \mathbb{Z}_{26} \to \mathbb{Z}_{26}$$
$$p_i \mapsto (a.p_i + b) \mod 26.$$

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- *Encrypt* ***COLLEGE*** *using* $a = 5$ *and* $b = 4$
- *Convert* ***C O L L E G E*** *in numeric form*

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Ring $(\mathbb{Z}_{26}, +, \cdot)$ in Affine Cipher

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Exercise

1. *Let* $f_{(a,b)}$ & $f_{(c,d)}$ *be two affine ciphers s/t*

$$f_{(a,b)}(x) \equiv (a.x + b) \mod 26$$
$$f_{(c,d)}(x) \equiv (c.x + d) \mod 26$$

Is $f_{(c,d)} \circ f_{(a,b)}$ *a stronger encryption scheme than* $f_{(a,b)}$*?*

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   Is $f_{(c,d)} \circ f_{(a,b)}$ *a stronger encryption scheme than* $f_{(a,b)}$*?*

2. *How many functions of type* $f_{(a,b)}$ *are there for affine cipher in* $\mathbb{Z}26$*?*

Rings

Theorem

In the ring $\mathbb{Z}_n$, the zero-divisors are precisely those non-zero elements that are not relatively prime to n.

Corollary

If p is prime, then $\mathbb{Z}_p$ has no zero-divisor

Theorem

The cancellation laws holds in a ring R iff R has no zero-divisor.

Homomorphism

Definition

A mapping ϕ from the ring R into the ring R' is said to be a homomorphism if

1. (i) $\phi(a+b) = \phi(a) + \phi(b)$
2. (ii) $\phi(a.b) = \phi(a).\phi(b)$

Definition

A mapping ϕ from the ring R into the ring R' is said to be a isomorphism if ϕ is a homomorphism as well as one-to-one and onto.

Homomorphism

Lemma

If ϕ is a homomorphism of R into R', then

(i) $\phi(0) = 0$

(ii) $\phi(-a) = -\phi(a)\ \forall\ a \in R$

Homomorphism

Lemma

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Definition

If ϕ is a homomorphism of R into R' then the kernel of phi, $I(\phi)$, is the set of all elements $a \in R$ s/t $\phi(a) = 0$, the zero-element of R'.

Homomorphism

Lemma

If ϕ is a homomorphism of R into R' with kernel $I(\phi)$, then

(i) *$I(\phi)$ is a subgroup of R under addition.*

(ii) *If $a \in I(\phi)$ and $r \in R$ then both $a.r, r.a \in I(\phi)$.*

Homomorphism

Lemma

If ϕ is a homomorphism of R into R' with kernel $I(\phi)$, then

- (i) $I(\phi)$ *is a subgroup of* R *under addition.*
- (ii) *If* $a \in I(\phi)$ *and* $r \in R$ *then both* $a.r, r.a \in I(\phi)$.

Example

Let $J(\sqrt{2})$ *be all real numbers of the form* $m + n\sqrt{2}$ *where* $m, n \in \mathbb{Z}$; $J(\sqrt{2})$ *forms a ring under the usual addition andmultiplication of real numbers. (Verify!)*

Define $\phi : J(\sqrt{2}) \to J(\sqrt{2})$ *by*

$$\phi(m + n\sqrt{2}) = m - n\sqrt{2}.$$

ϕ *is a homomorphism of* $J(\sqrt{2})$ *onto* $J(\sqrt{2})$ *and its kernel* $I(\phi)$, *consists only of 0. (Verify!)*

Ideals and Quotient Rings

Definition

A non-empty subset I of R is said to be a (two-sided) ideal of R if

(i) *I is a subgroup of R under addition.*

(ii) *For every $u \in I$ and $r \in R$, both $ur, \ \& \ ru \in I$.*

Ideals and Quotient Rings

Lemma

If I is an ideal of the ring R, then R/I is a ring and is a homomorphic image of R.

Ideals and Quotient Rings

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If I is an ideal of the ring R, then R/I is a ring and is a homomorphic image of R.

Proof.

Hint:

- R/I is the set of all the distinct cosets of I in R
- R/I consists of all the cosets $a + I$, where $a \in R$.
- R/I is automatically a group under addition $(a + I) + (b + I) = (a + b) + I$.
- Define the multiplication in R/I as $(a + I)(b + I) = ab + I$
- Define homomorphism $\phi : R \to R/I$ by $\phi(a) = a + I$ for every $a \in R$.
- Prove that kernel of ϕ is exactly I.

□

Ideals and Quotient Rings

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If I is an ideal of the ring R, then R/I is a ring and is a homomorphic image of R.

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- R/I consists of all the cosets $a+I$, where $a \in R$.
- R/I is automatically a group under addition $(a+I)+(b+I)=(a+b)+I$.
- Define the multiplication in R/I as $(a+I)(b+I)=ab+I$
- Define homomorphism $\phi : R \to R/I$ by $\phi(a)=a+I$ for every $a \in R$.
- Prove that kernel of ϕ is exactly I.

□

If R is commutative then so is R/I. If R has the identity element 1, then R/I has the identity $1+I$

Ideals and Quotient Rings

Theorem

Let R, R' be rings and ϕ be a homomorphism of R onto R' with kernel I. Then R' is isomorphic to R/I.

Moreover, there is a one-to-one correspondence between the set of ideals of R' and the set of ideals of R which contain I.

This correspondence can be achieved by associating with an ideal I' in R' the ideal I in R is defined by $I = \{x \in R \mid \phi(x) \in I'\}$. R/I is isomorphic to R'/I'.

Ideals and Quotient Rings

Lemma

Let R be a commutative ring with identity whose only ideals are (0) and R itself. Then R is a field.

Ideals and Quotient Rings

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Let R be a commutative ring with identity whose only ideals are (0) and R itself. Then R is a field.

Proof.

- Suppose that $a \neq 0$ is in R. Consider the set $Ra = \{xa \mid x \in R\}$.
- **Claim:** Ra is an ideal of R.
- Ra is an additive subgroup of R.
- If $r \in R,\ u \in Ra,\ ru = r(r_1a) = (rr_1)a \in Ra$. Ra is an ideal of R.
- $Ra = (0)$ *or* $Ra = R$. $\because 0 \neq a = 1a \in Ra,\ Ra \neq (0)$; thus, we have $Ra = R$.
- $\because 1 \in R$ so, it can be realized as a multiple of a; $\exists\ b \in R$ s/t $ba = 1$.

□

Ideals and Quotient Rings

Definition

*An ideal $M \neq R$ in a ring R is said to be a **maximal ideal** of R if whenever U is an ideal of R s/t $M \subset U \subset R$, then either $R = U$ or $M = U$.*

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An ideal $M \neq R$ in a ring R is said to be a ***maximal ideal*** *of R if whenever U is an ideal of R s/t $M \subset U \subset R$,then either $R = U$ or $M = U$.*

Exercise

Let $R = \mathbb{Z}$ be the ring of integers, and let U be an ideal of R. $\because U \leq R$ we know that $U = n_0\mathbb{Z}$; we write this as $U = (n_0)$. What values of n_0 lead to maximal ideals?

Ideals and Quotient Rings

Solution

- *First, we assume* p *is prime* $\Rightarrow P = (p)$ *is a maximal ideal of* R.
  - *If* U *is an ideal of* R *and* $P \subset U$, *then* $U = (n_0)$ *for some integer* n_0
  - $\because p \in P \subset U, p = mn_0$ for some $m \in \mathbb{Z}$
    $\because p$ is a prime $\Rightarrow n_0 = 1$ or $n_0 = p$
  - *If* $n_0 = p$, *then* $P \subset U = (n_0) \subset P, \Rightarrow U = P$
  - *If* $n_0 = 1$,*then* $1 \in U$, *hence* $r = 1r \in U \forall\ r \in R$ *whence* $U = R$

Ideals and Quotient Rings

Solution

- *Now, we assume $M = (n_0)$ is a maximal ideal of $R \Rightarrow n_0$ must be prime.*
  - ***Claim:** n_0 must be a prime*
  - *If $n_0 = ab$, where $a, b \in \mathbb{N}$, then $U = (a) \supset M$, hence $U = R$ or $U = M$.*
    - *If $U = R$, then $a = 1 \Rightarrow n_0$ is prime*
    - *If $U = M$, then $a \in M$ and so $a = rn_0$ for some integer r,*
      $\because$ every element of M is a multiple of n_0
    - *But then $n_0 = ab = rn_0b, \Rightarrow rb = 1$, so that $b = 1, n_0 = a$.*
      Thus, n_0 is a prime number.

Ideals and Quotient Rings

Example (Maximal Ideal)

Let R *be the ring of all the real-valued, continuous functions on the closed unit interval* $[0, 1]$.

Let

$$M = \{f(x) \in R \mid f(1/2) = 0\}.$$

M *is certainly an ideal of R. Moreover, it is a maximal ideal of* R.

Ideals and Quotient Rings

Theorem

If R is a commutative ring with identity and M is an ideal of R, then M is a maximal ideal of $R \iff R/M$ is a field.

Ideals and Quotient Rings

Theorem

If R is a commutative ring with identity and M is an ideal of R, then M is a maximal ideal of $R \iff R/M$ is a field.

Proof.

- Suppose, first, R/M is a field.
  - $\because R/M$ is a field its only ideals are (0) and R/M itself.
  - There is a one-to-one correspondence between the set of ideals of R/M and the set of ideals of R which contain M.
  - The ideal M of R corresponds to the ideal (0) of R/M whereas the ideal R of R corresponds to the ideal R/M of R/M in this one-to-one mapping.
  - Thus there is no ideal between M and R other than these two, whence M is a maximal ideal.

□

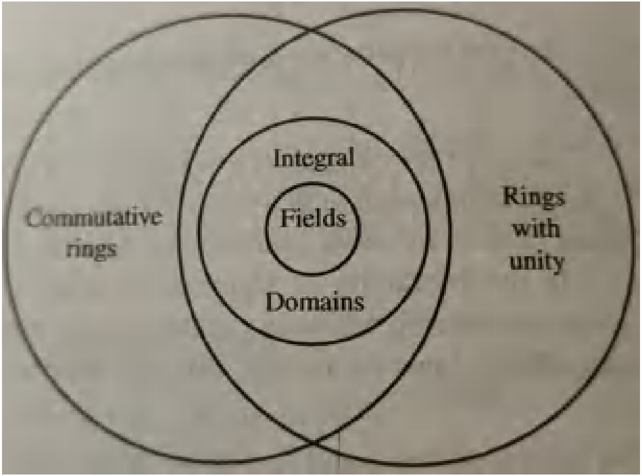
Ideals and Quotient Rings

Proof.

- Now, assume that M is a maximal ideal of R
  - $\because$ M is a maximal ideal of R, R/M has only (0) and itself as ideals.
  - Furthermore R/M is commutative with identity element since R enjoys both these properties.
  - By the lemma, we can say that R/M is a field.

$\square$

Ideals and Quotient Rings



The Field of Quotients of an ID

Definition

*A ring R can be **imbedded** in a ring R' if there is an isomorphism*[a] *of R into R'.*

*R' will be called an **over-ring** or **extension** of R if R can be imbedded in R'.*

---

[a] If R & R' have identity element, then this isomorphism takes 1 onto 1'.

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[a] If R & R' have identity element, then this isomorphism takes 1 onto 1'.

- Let D be our integral domain. Let a/b denotes all quotients where $a, b \in D$ and $b \neq 0$
- Define:
  - $\frac{a}{b} = \frac{c}{d} \iff ad = bc$
  - $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
  - $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$

The Field of Quotients of an ID

- $\mathcal{M} = \{(a, b) \mid a, b \in D \;\&\; b \neq 0\}$
- Define a relation on $\mathcal{M}$ as follows:

$$(a, b) \sim (c, d) \iff ad = bc.$$

- Prove that $\sim$ is an equivalence relation on $\mathcal{M}$
- Let $[a, b]$ be the equivalence class in $\mathcal{M}$ of (a, b).
- Let F be the set of all such equivalence classes $[a, b]$ where $a, b \in D$ and $b \neq 0$.
- Prove that F is a field where

$$[a, b]^{-1} = [b, a], \because \; a \neq 0$$

The Field of Quotients of an ID

Theorem

Every integral domain can be imbedded in a field.

Euclidean Rings

Definition

An integral domain R *is said to be a* *Euclidean ring* *if for every* $a \neq 0$ *in* R *there is defined a non-negative integer* $d(a)$ *s/t*

(i) $\forall\ a, b \in R$, both non-zero, $d(a) \leq d(ab)$.

(ii) For any $a, b \in R$, both non-zero, $\exists\ q, r \in R$ s/t $a = qb + r$ where either $r = 0$ or $d(r) < d(b)$.

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An integral domain R is said to be a Euclidean ring if for every $a \neq 0$ in R there is defined a non-negative integer $d(a)$ s/t

1. (i) $\forall\ a, b \in R$, *both non-zero,* $d(a) \leq d(ab)$.
2. (ii) *For any* $a, b \in R$, *both non-zero,* $\exists\ q, r \in R$ *s/t* $a = qb + r$ *where either* $r = 0$ *or* $d(r) < d(b)$.

Note:

- We do not assign a value to $d(0)$.
- $d(a) =$ absolute value of a acts as the required function.

Euclidean Rings

Theorem

Let R be a Euclidean ring and let A be an ideal of R. Then $\exists\ a_0 \in A$ s/t A consists exactly of all a_0x as x ranges over R.

Euclidean Rings

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Let R be a Euclidean ring and let A be an ideal of R. Then $\exists\ a_0 \in A$ s/t A consists exactly of all a_0x as x ranges over R.

Proof.

- If A just consists of the element 0, put $a_0 = 0$
- Thus, we assume that there is an $a \neq 0$ in A.
- Pick an $a_0 \in A$ s/t $d(a_o)$ is minimal.
- $\because a \in A$, by the properties of Euclidean rings there exist $q, r \in R$ s/t $a = qa_0 + r$ where $r = 0$ or $d(r) < d(a_0)$.
- $\because a_0 \in A$ and A is an ideal of $R, qa_0 \in A$.

  $\Rightarrow a - qa_0 \in A$; but $r = a - qa_0$, whence $r \in A$.
- If $r \neq 0$ then $d(r) < d(a_0)$, giving us an element $r \in A$ whose d-value is smaller than that of a_0, in contradiction to our choice of $a_0 \in A$ of minimal d-value.

□

Euclidean Rings

Definition

An integral domain R with identity is a ***principal ideal ring*** *if every ideal A in R is of the form $A = (a)$ for some $a \in R$, where the notation $(a) = \{xa \mid x \in R\}$ to represent the ideal of all multiples of a.*

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Exercise

A Euclidean ring possesses the identity element.

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Exercise

A Euclidean ring possesses the identity element.

Definition

If $a \neq 0$ and b are in a commutative ring R then a is said to divide b if $\exists$ a $c \in R$ s/t $b = ac$. We shall use the symbol $a \mid b$ to represent the fact that a divides b and $a \nmid b$ to mean that a does not divide b.

Euclidean Rings

Definition

If $a, b \in R$ *then* $d \in R$ *is said to be a greatest common divisor of* a *and* b *if*

(i) $d \mid a$ & $d \mid b$.

(ii) *Whenever* $c \mid a$ *and* $c \mid b$ *then* $c \mid d$.

Euclidean Rings

Definition

If $a, b \in R$ then $d \in R$ is said to be a greatest common divisor of a and b if

1. (i) $d \mid a \;\&\; d \mid b$.
2. (ii) *Whenever $c \mid a$ and $c \mid b$ then $c \mid d$.*

Lemma

Let R be a Euclidean ring. Then any two elements $a \;\&\; b \in R$ have a greatest common divisor d. Moreover $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$.

Euclidean Rings

Proof.

- Let $A = \{ra + sb \ : \ r, s \in R\}$
- Prove that A is an ideal of R.
- Since A is an ideal of R, $\therefore A$ is principle ideal ring.
- $\exists\, d \in A$ s/t every element in A is a multiple of d.
- $\because R$ is a Euclidean ring, R contains identity.
- Thus, $a = 1.a + 0.b \in A, \ b = 0.a + 1.b \in A$
- They are both multiples of d, whence $d \mid a \ \& \ d \mid b$.
- Finally, suppose that $c \mid a \ \& \ c \mid b$; then $c \mid \lambda a + \mu b = d$.

□

Euclidean Rings

Definition

Let R be a commutative ring with identity. An element $a \in R$ is a ***unit*** *in R if $\exists$ an element $b \in R$ s/t $ab = 1$.*

Do not confuse a ***unit*** *with a* ***unit element****. A unit in a ring is an element whose inverse is also in the ring.*

Exercise

Let R be an integral domain with identity and suppose that for $a, b \in R$ both $a \mid b, \ \& \ b \mid a$. Then $a = ub$, where u is a unit in R.

Definition

Let R be a commutative ring with identity. Two elements $a \ \& \ b \in R$ are said to be associates if $b = ua$ for some unit $u \in R$.

Euclidean Rings

Definition

In the Euclidean ring R a nonunit π is said to be a prime element of R if whenever $\pi = ab$, where $a, b \in R$, then one of a or b is a unit in R.

Lemma

Let R be a Euclidean ring. Then every element in R is either a unit in R or can be written as the product of a finite number of prime elements of R.

Definition

In the Euclidean ring R, a & $b \in R$ are said to be relatively prime if $\gcd(a, b)$ is a unit of R.

Euclidean Rings

Lemma

Let R be a Euclidean ring. Suppose that for $a, b, c \in R,\ a \mid bc$ but $\gcd(a, b) = 1$. Then $a \mid c$.

Lemma

If π is a prime element in the Euclidean ring R and $\pi \mid ab$ where $a, b \in R$ then π divides at least one of a or b.

Theorem (Unique Factorization Theorem)

Let R be a Euclidean ring and $a \neq 0$ a nonunit in R. Suppose that

$$a = \pi_1 \pi_2 \ldots \pi_n = \pi'_1 \pi'_2 \ldots \pi'_m,$$

where the π_i & π'_j are prime elements of R. Then $n = m$ and each $\pi_i,\ 1 \leq i \leq n$ is an associate of some $\pi'_j,\ 1 \leq j \leq m$ and conversely each π'_k is an associate of some π_q.

Euclidean Rings

Lemma

Let R *be a Euclidean ring. Suppose that for* $a, b, c \in R,\ a \mid bc$ *but* $\gcd(a, b) = 1$*. Then* $a \mid c$*.*

Lemma

If π *is a prime element in the Euclidean ring* R *and* $\pi \mid ab$ *where* $a, b \in R$ *then* π *divides at least one of* a *or* b*.*

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Euclidean Rings

Every nonzero element in a Euclidean ring R can be uniquely written (up to associates) as a product of prime elements or is a unit in R.

Euclidean Rings

Every nonzero element in a Euclidean ring R can be uniquely written (up to associates) as a product of prime elements or is a unit in R.

Lemma

The ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R iff a_0 is a prime element of R.

Polynomial Rings

- Let F be a field. By the ring of polynomials in the indeterminate, x, denoted by $F[x]$,

$$F[x] = \{a_0 + a_1 x + \ldots + a_n x^n, \; : \; n \in \mathbb{N} \;\&\; a_i \in \mathbb{F}, \; for\; 0 \leq i \leq n\}.$$

Exercise

$F[x]$ is an integral domain, when F is a field (integral domain)

Theorem

$F[x]$ is a Euclidean ring, when F is a field (Euclidean domain)

Polynomial Rings

Lemma

$F[x]$ is a principal ideal ring, when F is a field

Lemma

Given two polynomials $f(x), g(x) \in F[x]$ and let $d(x) = \gcd(f(x), g(x))$. Then $d(x)$ can be expressed as

$$d(x) = \lambda(x)f(x) + \mu(x)g(x).$$

Polynomial Rings

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$$d(x) = \lambda(x)f(x) + \mu(x)g(x).$$

Definition

A polynomial $p(x) \in F[x]$ is said to be irreducible over F if whenever $p(x) = a(x)b(x)$ with $a(x), b(x) \in F[x]$, then one of $a(x)$ or $b(x)$ has degree 0 (i.e., is a constant).

Polynomial Rings

Lemma

Any polynomial in $F[x]$ can be written in a unique manner as a product of irreducible polynomials in $F[x]$.

Lemma

The ideal $A = (p(x))$ in $F[x]$ is a ***maximal ideal*** *iff $p(x)$ is irreducible over F.*

Polynomial Rings

Lemma

Any polynomial in $F[x]$ can be written in a unique manner as a product of irreducible polynomials in $F[x]$.

Lemma

*The ideal $A = (p(x))$ in $F[x]$ is a **maximal ideal** iff $p(x)$ is irreducible over F.*

Definition

The polynomial $f(x) = a_0 + a_1x + \ldots + a_nx^n$, where the $a_0, a_1, a_2, \ldots,$ are integers is said to be primitive if the greatest common divisor of $a_0, a_1, \ldots, a_n$ is 1.

Polynomial Rings

Definition

The content of the polynomial $f(x) = a_0 + a_1x + \ldots + a_nx^n$, where the a_i's are $\in \mathbb{Z}$, is the greatest common divisor of the integers $a_0, a_1, \ldots, a_n$.

Theorem

If the primitive polynomial $f(x)$ can be factored as the product of two polynomials having rational coefficients, it can be factored as the product of two polynomials having integer coefficients.

Polynomial Rings

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The content of the polynomial $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, where the a_i's are $\in \mathbb{Z}$, is the greatest common divisor of the integers $a_0, a_1, \ldots, a_n$.

Theorem

If the primitive polynomial $f(x)$ can be factored as the product of two polynomials having rational coefficients, it can be factored as the product of two polynomials having integer coefficients.

Definition

A polynomial is said to be integer monic if all its coefficients are integers and its highest coefficient is 1.

Polynomial Rings

Theorem (THE EISENSTEIN CRITERION)

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ *be a polynomial with integer coefficients. Suppose that for some prime number* $p, p \nmid a_n, p \mid a_0, p \mid a_1, p \mid a_2, \ldots, p \mid a_{n-1}, p^2 \nmid a_0$*. Then* $f(x)$ *is irreducible over the rationals.*

Polynomial Rings

Lemma

If R is an integral domain, then so is $R[x]$.

Polynomial Rings

Lemma

If R is an integral domain, then so is $R[x]$.

Definition

An element a which is not a unit in R will be called irreducible (or a prime element[a]) if, whenever $a = bc$ with $b, c \in R$, then one of b or c must be a unit in R.

[a] in case of R is a UFD

Polynomial Rings

Definition

An integral domain, R, with identity element is a unique factorization domain (UFD) if any nonzero element in R is either a unit or can be written as the product of a finite number of irreducible elements of R and the the decomposition is unique up to the order and associates of the irreducible elements.

Polynomial Rings

Definition

An integral domain, R, with identity element is a unique factorization domain (UFD) if any nonzero element in R is either a unit or can be written as the product of a finite number of irreducible elements of R and the the decomposition is unique up to the order and associates of the irreducible elements.

Lemma

If R is a unique factorization domain and if $a, b \in R$, then a and b have a greatest common divisor $(a, b) \in R$.

Polynomial Rings

Lemma

If R is a unique factorization domain, then the product of two primitive polynomials in $R[x]$ is again a primitive polynomial in $R[x]$.

Lemma

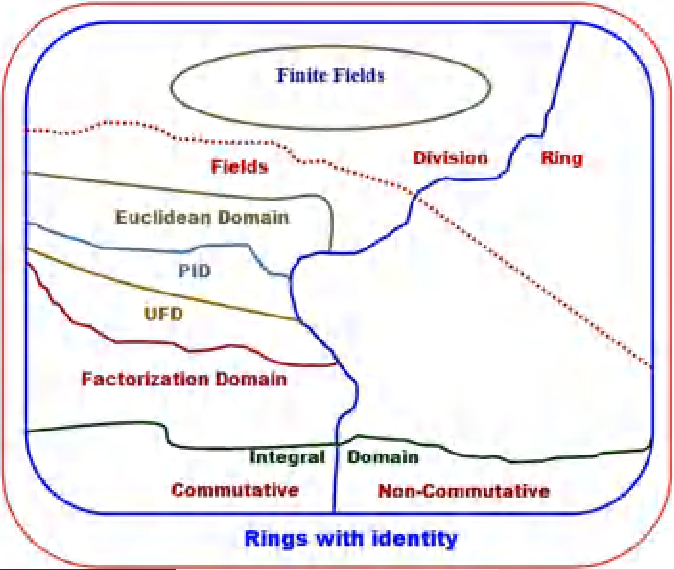
If R is a unique factorization domain and if $p(x)$ is a primitive polynomial in $R[x]$, then it can be factored in a unique way as the product of irreducible elements in $R[x]$.

Polynomial Rings

Theorem

If R is a unique factorization domain, then so is $R[x]$.

Ring Structure



Outline

Vector Spaces

Definition

A non-empty set $\mathbf{V}$ *is said to be a vector space over a field* $\mathbb{F}$, *is denoted by* $(\mathbf{V}, +, \cdot, \mathbb{F})$ *if* $\mathbf{V}$ *is an abelian group under an operation which we denote by* $+$, *and if for every* $\alpha \in \mathbb{F},\ v \in \mathbf{V}$ *there is defined an element, written* $\alpha v \in \mathbf{V}$ *subject to*

(i) $\alpha.(v + w) = \alpha.v + \alpha.w$;

(ii) $(\alpha + \beta).v = \alpha.v + \beta.v$;

(iii) $\alpha.(\beta.v) = (\alpha.\beta).v$;

(iv) $1.v = v$;

or all $\alpha, \beta \in \mathbb{F},\ v, w \in \mathbf{V}$ *(where the* 1 *represents the identity element of* $\mathbb{F}$ *under multiplication).*

Linear Independence and Bases

Definition

If $\mathbf{V}$ is a vector space over $\mathbb{F}$ and if $v_1, \ldots, v_n \in \mathbf{V}$ then any element of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n,$$

*where the $\alpha_i \in \mathbb{F}$, is a **linear combination** of $v_1, \ldots, v_n$ over $\mathbb{F}$.*

Linear Independence and Bases

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*where the $\alpha_i \in \mathbb{F}$, is a **linear combination** of $v_1, \ldots, v_n$ over $\mathbb{F}$.*

Definition

*If S is a nonempty subset of the vector space $\mathbf{V}$, then $L(S)$, the **linear span** of S, is the set of all linear combinations of finite sets of elements of S.*

Linear Independence and Bases

Lemma

$L(S)$ *is a subspace of* $\mathbf{V}$.

Linear Independence and Bases

Lemma

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Definition

If $\mathbf{V}$ *is a vector space and if* $v_1, \ldots, v_n$ *are in* $\mathbf{V}$*, we say that they are* ***linearly dependent*** *over* $\mathbb{F}$ *if there exist elements* $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$*, not all of them* *0**, s/t*

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0.$$

If the vectors $v_1, \ldots, v_n$ *are not linearly dependent over* $\mathbb{F}$*, they are said to be* ***linearly independent*** *over* $\mathbb{F}$.

Linear Independence and Bases

Lemma

If $v_1, \ldots, v_n \in \mathbf{V}$ are linearly independent, then every element in their linear span has a ! representation in the form $\lambda_1 v_1 + \ldots + \lambda_n v_n$ with the $\lambda_i \in \mathbb{F}$.

Theorem

If $v_1, \ldots, v_n$ are in $\mathbf{V}$ then either they are linearly independent or some v_k is a linear combination of the preceding ones, $v_1, \ldots, v_{k-1}$.

Corollary

If $\mathbf{V}$ is a finite-dimensional vector space, then it contains a finite set $v_1, \ldots, v_n$ of linearly independent elements whose linear span is $\mathbf{V}$.

Linear Independence and Bases

Definition

A subset S o f a vector space $\mathbf{V}$ is called a ***basis*** *of $\mathbf{V}$ if S consists of linearly independent elements*[a] *and $\mathbf{V} = L(S)$.*

[a] Any finite number of elements in S is linearly independent

Corollary

If $\mathbf{V}$ is a finite-dimensional vector space and if $u_1, \ldots, u_m$ span $\mathbf{V}$ then some subset of $u_1, \ldots, u_m$ forms a basis of $\mathbf{V}$.

Corollary

If $\mathbf{V}$ is finite-dimensional over $\mathbb{F}$ then any two bases of $\mathbf{V}$ have the same number of elements.

Linear Independence and Bases

Corollary

If $\mathbf{V}$ is finite-dimensional over $\mathbb{F}$ then $\mathbf{V}$ is isomorphic to $\mathbb{F}^{(n)}$ for a unique integer n; in fact, n is the number of elements in any basis of $\mathbf{V}$ over $\mathbb{F}$.

Definition

The integer n in the above Corollary is called the ***dimension*** *of $\mathbf{V}$ over $\mathbb{F}$.*

Outline

Field Extension

Definition

If $\mathbb{K}$ is a subfield of a field $\mathbb{M}$, then $\mathbb{M}$ is called an ***extension of the field*** *$\mathbb{K}$.*

Definition

Let $\mathbb{M}$ be an extension of a field $\mathbb{K}$. An element $u \in \mathbb{M}$ is said to be ***algebraic*** *over $\mathbb{K}$ if u satisfies a polynomial over $\mathbb{K}$ i.e., if elements $c_0, c_1, \ldots, c_n$ not all zero exit in $\mathbb{K}$ such that*

$$c_0 + c_1.u + \ldots + c_n.u^n = 0.$$

Field Extension

Definition

*An element of $\mathbb{M}$ which is not algebraic is said to be **transcendental** over $\mathbb{K}$.*

Definition

*An extension of a field $\mathbb{K}$ is called an **algebraic extension** if every member of it is algebraic over $\mathbb{K}$. Otherwise if $\exists$ a single element in the extension which is transcendental over $\mathbb{K}$, the extension is called a **transcendental** extension of $\mathbb{K}$.*

Extension as a Vector Space

- An extension $\mathbb{M}$ of a field $\mathbb{K}$ can be looked upon as a vector space over $\mathbb{K}$.

Extension as a Vector Space

- An extension $\mathbb{M}$ of a field $\mathbb{K}$ can be looked upon as a vector space over $\mathbb{K}$.
- $\because$ $\mathbb{M}$ is a field, $\therefore$ it is already an additive commutative group.
- Now the product of an element of $\mathbb{K}$ and an element of an element of $\mathbb{M}$ is a product of two elements of $\mathbb{M}$ and is therefore an element of $\mathbb{M}$.
- Hence, $\mathbb{M}$ is a vector space over $\mathbb{K}$.

Definition

If $\mathbb{M}$ is an extension of a field $\mathbb{K}$, then $\mathbb{M}$ may be looked upon as a vector space over $\mathbb{K}$. The dimension of this vector space is called the ***degree of the extension****, and is denoted by $[\mathbb{M} : \mathbb{K}]$.*

Extension as a Vector Space

Theorem (Paul Halmos)

Any finite extension of a field is an algebraic extension of the field.

Extension as a Vector Space

Theorem (Paul Halmos)

Any finite extension of a field is an algebraic extension of the field.

Proof.

- Let $\mathbb{M}$ be a finite extension of a field $\mathbb{K}$ and $[\mathbb{M} : \mathbb{K}] = n$.
- Then for any $u \in \mathbb{M}$, the $(n+1)$ elements $1, u, \ldots, u^n$ must be linearly dependent over $\mathbb{K}$.
- Hence, elements $c_0, c_1, \ldots, c_n$, not all zero exists in $\mathbb{K}$ such that

$$c_0.1 + c_1.u + \cdots + c_n u^n = 0.$$

- This shows that u is an algebraic over $\mathbb{K}$; but u was an arbitrary element of $\mathbb{M}$.
- Thus, it is proved that $\mathbb{M}$ is an algebraic extension of $\mathbb{K}$.

□

Extension as a Vector Space

Exercise

If $\mathbb{M}$ *is an extension of a field* $\mathbb{K}$ *and* $[\mathbb{M} : \mathbb{K}] = 1$, *show that* $\mathbb{M} = \mathbb{K}$.

Extension as a Vector Space

Exercise

If $\mathbb{M}$ *is an extension of a field* $\mathbb{K}$ *and* $[\mathbb{M} : \mathbb{K}] = 1$, *show that* $\mathbb{M} = \mathbb{K}$.

Solution

- $\because [\mathbb{M} : \mathbb{K}] = 1, \therefore$ *for any* $u \in \mathbb{M},\ 1\ \&\ u$ *must be linearly dependent over* $\mathbb{K}$.
- *Hence,* $\exists\ c_0\ \&\ c_1$ *not both zero in* $\mathbb{K}$ *s/t* $c_0.1 + c_1.u = 0$
  Clearly $c_1 \neq 0$, *[*$\because c_1 = 0$ *gives* $c_0 = 0$ *]*
- *Now,* $\because [\mathbb{M} : \mathbb{K}] = 1$ *is finite,* $\therefore$ *every elements of* $\mathbb{M}$ *is algebraic.*
- $\because \mathbb{K}$ *is a field and* $c_1 \neq 0, \therefore c_1^{-1}$ *exists in* $\mathbb{K}$.
- *Now from above equation we see that* $u = -c_1^{-1}c_0 \in \mathbb{K}$ *[*$\because\ \mathbb{K}$ *is a field]*
- $\because u$ *is arbitrary, therefore* $\mathbb{M} \subseteq \mathbb{K}$ *and* $\because \mathbb{M}$ *is an extension of a field* $\mathbb{K}$, $\therefore \mathbb{K} \subseteq \mathbb{M}$.
  Hence we have $\mathbb{M} = \mathbb{K}$.

Extension as a Vector Space

Theorem (Transitivity of Finite Extensions)

If $\mathbb{B}, \mathbb{C}$ & $\mathbb{D}$ *are 3 fields s/t* $\mathbb{B}$ *is a finite extension of* $\mathbb{C}$ *and* $\mathbb{C}$ *is finite extension of* $\mathbb{D}$, *then* $\mathbb{B}$ *is finite extension of* $\mathbb{D}$, *and* $[\mathbb{B} : \mathbb{D}] = [\mathbb{B} : \mathbb{C}] \times [\mathbb{C} : \mathbb{D}]$.

Extension as a Vector Space

Theorem (Transitivity of Finite Extensions)

If $\mathbb{B}, \mathbb{C}$ & $\mathbb{D}$ are 3 fields s/t $\mathbb{B}$ is a finite extension of $\mathbb{C}$ and $\mathbb{C}$ is finite extension of $\mathbb{D}$, then $\mathbb{B}$ is finite extension of $\mathbb{D}$, and $[\mathbb{B} : \mathbb{D}] = [\mathbb{B} : \mathbb{C}] \times [\mathbb{C} : \mathbb{D}]$.

Proof.

- Let $[\mathbb{B} : \mathbb{C}] = m$ & $[\mathbb{C} : \mathbb{D}] = n$. Let $\{u_1, \ldots, u_m\}$ be a basis of $\mathbb{B}$ over $\mathbb{C}$ & $\{v_1, \ldots, v_n\}$ be a basis of $\mathbb{C}$ over $\mathbb{D}$.
- Then any $t \in \mathbb{B}$ is of the form $t = \sum_{i=1}^{n} b_i u_i$, for certain elements $b_1, \ldots, b_m \in C$.
- $\because b_1, \ldots, b_m \in \mathbb{C}$ each of them is a linear combination of $\{v_1, \ldots, v_n\}$ with coefficient from $\mathbb{D}$.
- Let $b_i = \sum_{j=1}^{n} c_{ij} v_j$, where c_{ij}'s $\in \mathbb{D}$. But then $t = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_{ij} v_j \right) u_i = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} v_j u_i$

□

Extension as a Vector Space

Proof.

- This shows that the mn elements $v_j u_i$ generate $\mathbb{B}$ over $\mathbb{D}$.
- We show that these elements are independent over $\mathbb{D}$. For this, let $\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}v_j u_i = 0$. This can be written as $\sum_{i=1}^{m}\left(\sum_{j=1}^{n} d_{ij}v_j\right)u_i = 0$.
- Since u vectors are independent over $\mathbb{C}$ we get $\sum_{j=1}^{n} d_{ij}v_j = 0$, for $1 \le i \le m$.
- However, v vectors are independent over $\mathbb{D}$ we get $d_{ij} = 0$, for $1 \le i \le m$ & $1 \le j \le n$.
- Hence, the mn vectors $v_j u_i$ are indeed independent over $\mathbb{D}$ showing that these vectors form a basis of $\mathbb{B}$ over $\mathbb{D}$.
- Hence, $[\mathbb{B} : \mathbb{D}] = mn$ and thus $[\mathbb{B} : \mathbb{D}] = [\mathbb{B} : \mathbb{C}] \times [\mathbb{C} : \mathbb{D}]$.

□

Extension as a Vector Space

Exercise

If $\mathbb{B}$ is a finite extension of a field $\mathbb{D}$ and $\mathbb{C}$ is a field intermediate between $\mathbb{B}$ and $\mathbb{D}$, show that $\mathbb{B}$ is a finite extension of $\mathbb{C}$ and $\mathbb{C}$ is a finite extension of $\mathbb{D}$.

Corollary

If $[\mathbb{B} : \mathbb{C}] = p$, a prime number then there cannot be any field properly in between $\mathbb{B}$ and $\mathbb{C}$.

Exercise

1. *If $\mathbb{B}$ and $\mathbb{C}$ are finite extension of a field $\mathbb{D}$ and $\mathbb{D} \subset \mathbb{C} \subset \mathbb{B}$, then show that $\mathbb{B}$ is a finite extension of $\mathbb{D}$.*
2. *If $\mathbb{B}$ is a finite extension of a field $\mathbb{D}$ and $\mathbb{C}$ is a subfield of $\mathbb{B}$ then show that $[\mathbb{C} : \mathbb{D}]$ divides $[\mathbb{B} : \mathbb{D}]$*
3. *The field of complex numbers $\mathbb{C}$ is a finite extension of degree 2 over the real field $\mathbb{R}$.*

Adjunction

- Let $\mathbb{M}$ be an extension of a field $\mathbb{K}$ and let $G \subset \mathbb{M}$.
- Then the intersection of all subfields of $\mathbb{M}$ containing $\mathbb{K}$ and G is the smallest subfield of $\mathbb{M}$ containing $\mathbb{K}$ and G.
- This subfield is denoted by $\mathbb{K}(G)$ and is called the subfield of $\mathbb{M}$ obtained from $\mathbb{K}$ by the **adjunction** of the subset G or simply '$\mathbb{K}$ adjunction G'.
- If G is a finite set equal to $\{a_1, \ldots, a_n\}$ then $\mathbb{K}(G)$ is also written as $\mathbb{K}(a_1, \ldots, a_n)$.

Adjunction

Theorem

If $\mathbb{M}$ is a finite extension of a field $\mathbb{K}$, then $\mathbb{M}$ can be obtained by adjoining a finite number of elements $u_1, \ldots, u_m$ to $\mathbb{K}$ so that $\mathbb{M} = \mathbb{K}(u_1, \ldots, u_m)$ where $u_1, \ldots, u_m$ are algebraic over $\mathbb{K}$.

Adjunction

Theorem

If $\mathbb{M}$ is a finite extension of a field $\mathbb{K}$, then $\mathbb{M}$ can be obtained by adjoining a finite number of elements $u_1, \ldots, u_m$ to $\mathbb{K}$ so that $\mathbb{M} = \mathbb{K}(u_1, \ldots, u_m)$ where $u_1, \ldots, u_m$ are algebraic over $\mathbb{K}$.

Proof.

- $\because$ $\mathbb{M}$ is a finite extension of $\mathbb{K}$ each element of $\mathbb{M}$ is algebraic over $\mathbb{K}$.
- If $\mathbb{M} = \mathbb{K}$ the theorem is vacuously true.
- If $\mathbb{M} \neq \mathbb{K}$ then $\exists$ at least one element $u_1 \in \mathbb{M} \setminus \mathbb{K}$. If $\mathbb{M} = \mathbb{K}(u_1)$ the theorem is proved.
- If $\mathbb{M} \neq \mathbb{K}(u_1)$, $\exists$ at least one element $u_2 \in \mathbb{M} \setminus \mathbb{K}(u_1)$. If $\mathbb{M} = \mathbb{K}(u_1, u_2)$ the theorem is proved.
- If not, we carry on the process and after a finite number of steps we shall arrive at an extension $\mathbb{K}(u_1, \ldots, u_m)$ s/t $\mathbb{M} = \mathbb{K}(u_1, \ldots, u_m)$. $\because$ at each step we arrive at proper extension of the previous one and thus an extension ≥ 2; but $\mathbb{M}$ is of finite degree over $\mathbb{K}$.

Adjunction

Definition

Let $\mathbb{M}$ be an extension of a field $\mathbb{K}$ and u be any element of $\mathbb{M}$. Then the field $\mathbb{K}(u)$ obtained from $\mathbb{K}$ by adjunction of the single element u is called a ***simple extension of*** $\mathbb{K}$.

The extension is called a ***simple algebraic extension*** *or a* ***simple transcendental extension*** *according as u is algebraic or transcendental over $\mathbb{K}$.*

Definition

Let $\mathbb{M}$ be an extension of a field $\mathbb{K}$ and $u \in \mathbb{M}$ be algebraic over $\mathbb{K}$. Then the monic polynomial of the least degree over $\mathbb{K}$ satisfied by u is called the ***minimal polynomial*** *of u over $\mathbb{K}$. If $f(x)$ is the minimal polynomial of u over $\mathbb{K}$, then degree of $(f(x))$ is also called the degree of u over $\mathbb{K}$, written as $deg(u)$ over $\mathbb{K}$.*

Adjunction

Exercise

If p is a prime and $\mathbb{Q}$ the rational field, then show that $\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} \;:\; a, b \in \mathbb{Q}\}$

Adjunction

Exercise

If p is a prime and $\mathbb{Q}$ the rational field, then show that $\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} \ : \ a, b \in \mathbb{Q}\}$

Solution

- *Let* $\alpha = \sqrt{p}$. *Then* $\alpha^2 = p$ *i.e.,* $\alpha^2 - p = 0$.
- *Thus,* $\alpha = \sqrt{p}$ *satisfies the polynomial* $x^2 - p$ *over* $\mathbb{Q}$. *But* $\sqrt{p}$ *can't satisfy a polynomial of degree* < 2 *i.e., a polynomial of degree 1 over* $\mathbb{Q} \because \sqrt{p} \notin \mathbb{Q}$.
- *Hence,* $deg \sqrt{p}$ *over* $\mathbb{Q} = 2$.
- *Thus,* $\{1, \sqrt{p}\}$ *forms a basis of* $\mathbb{Q}(\sqrt{p})$ *over* $\mathbb{Q}$.
- *Hence, any number of* $\mathbb{Q}(\sqrt{p})$ *is of the form* $a.1 + b.\sqrt{p}$ *where* $a, b \in \mathbb{Q}$.

Adjunction

Exercise

Find the inverse of $5u + 6$ as a polynomial in u over the rationals given that the minimal polynomial of u over the rationals is $x^2 + 7x - 11$.

Adjunction

Exercise

Find the inverse of $5u + 6$ as a polynomial in u over the rationals given that the minimal polynomial of u over the rationals is $x^2 + 7x - 11$.

Solution

We have $u^2 + 7u - 11 = 0$ *or* $u^2 = -7u + 11$.
Let $au + b$ *be the required inverse of* $5u + 6$.
We must have
$$\begin{aligned} 1 &= (5u + 6)(au + b) \\ &= 5au^2 + (6a + 5b)u + 6b \\ &= 5a(-7u + 11) + (6a + 5b)u + 6b \\ &= (-29a + 5b)u + (55a + 6b) \end{aligned}$$
So, we have $-29a + 5b = 0 \quad \& \quad 55a + 6b = 1$
Therefore the required inverse is $\frac{5}{449}u + \frac{29}{449}$

Algebraic Closure

Definition

Let $\mathbb{M}$ be an extension of a field $\mathbb{K}$. Then the set $\mathbb{E}$ of all elements of $\mathbb{M}$ which are algebraic over $\mathbb{K}$ is a subfield of $\mathbb{M}$ containing $\mathbb{K}$. This field $\mathbb{E}$ is called the ***algebraic closure*** *of $\mathbb{K}$ in $\mathbb{M}$.*

Definition

Let $\mathbb{K}$ be any field. Then an algebraic extension $\bar{\mathbb{K}}$ is said to be ***algebraic closure*** *iff $\bar{\mathbb{K}}$ is algebrically closed over $\mathbb{K}$.*

Note 1: If $\mathbb{F}$ is an algebraically closed field, then the algebraic closure of $\mathbb{F}$ is $\mathbb{F}$ itself.

Note 2: (**Fundamental Theorem of Algebra**) The complex field $\mathbb{C}$ is algebraically closed.

Finite Fields

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- If $\mathbb{F}$ is a finite field, then $\mathbb{F}$ contains p^m elements for some prime p and integer $m \geq 1$.
- For every prime power order p^m, there is a $!$ finite field of order p^m. This field is denoted by $\mathbb{F}_{p^m}$, or sometimes by $GF(p^m)$.
- For $m = 1$, $\mathbb{F}_p$ or $GF(p)$ is a field. If p is a prime then $\mathbb{Z}_p$ is a field.

$$\mathbb{F}_p \cong GF(p) \cong \mathbb{Z}_p.$$

Finite Fields

- Let $\mathbb{F}_q$ be a finite field of order $q = p^m$.
  - Then every subfield of $\mathbb{F}_q$ has order p^n, for some n which is a positive divisor of m.
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- A generator of the cyclic group $\mathbb{F}_q^*$ is called a **primitive element** or **generator** of $\mathbb{F}_q$.

Finite Fields

Subfields of $\mathbb{F}_{2^{30}}$ and their relation:

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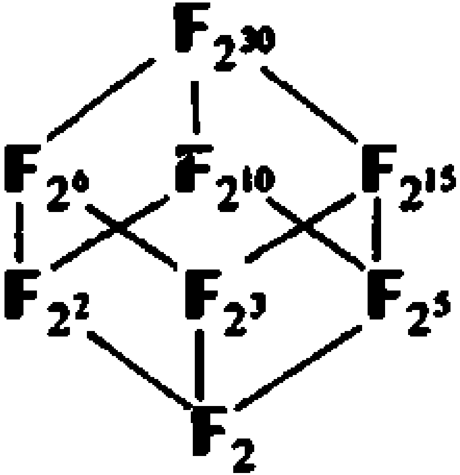

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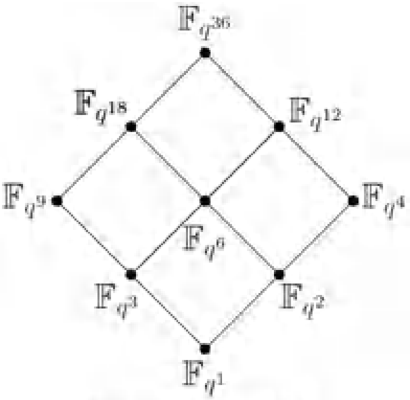

The subfields of $\mathbb{F}_{q^{36}}$

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- The ideal $< f(x) >$ is a maximal ideal.
- Then $Z_p[x]/ < f(x) >$ is a finite field of order p^m.
- For each $m \geq 1$, $\exists$ a monic irreducible polynomial of degree m over $\mathbb{Z}_p$.

  Hence, every finite field has a polynomial basis representation.

Construction of Finite Field of Order p^m

Theorem

The number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n is given by

$$\frac{1}{n}\sum_{d|n}\mu(d)q^{n/d},$$

where μ is Möbius function.

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Definition

TheMöbius function μ is the function on $\mathbb{N}$ defined by

$$\mu(n) = \begin{cases} 1 & if\ n = 1, \\ (-1)^k & if\ n\ is\ the\ product\ of\ k\ distinct\ primes, \\ 0 & if\ n\ is\ divisible\ by\ square\ of\ a\ prime. \end{cases}$$

Construction of Finite Field of Order 2^4

(i) First consider α is a root of the irreducible polynomial $x^4 + x + 1$ over $GF(2)$

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(iii) Now Consider the irreducible polynomial $x^4 + x^3 + x^2 + x + 1$ or $x^4 + x^3 + 1$ over $GF(2)$.

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Algorithm

Input: a non-zero polynomial $g(x) \in \mathbb{F}_{p^m}$[a].

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2. *Return*($s(x)$).

[a] The elements of the field $\mathbb{F}_{p^m}$ are represented as $\mathbb{Z}_p[x]/ < f(x) >$, where $f(x) \in \mathbb{Z}_p[x]$ is an irreducible polynomial of degree m over $\mathbb{Z}_p$.

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An irreducible polynomial $f \in \mathbb{Z}_p[x]$ *of degree* m *is called a* ***primitive polynomial*** *if* α *is a generator of* $\mathbb{F}^*_{p^m}$*, the multiplicative group of all the non-zero elements in* $\mathbb{F}_{p^m} = \mathbb{Z}_p[x]/ < f(x) >$*, where* α *is a root of the polynomial* $f(x)$ *over its extension field.*

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Example

- **Addition (in the field $GF(2^8)$)**

  The sum of two elements is the polynomial with coefficients that are given by the sum modulo 2 of the coefficients of the two terms.

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$$(x^6 + x^4 + x^2 + x + 1) + (x^7 + x + 1) = x^7 + x^6 + x^4 + x^2 = D4$$

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- **Multiplication**
  Multiplication in $GF(2^8)$ corresponds with multiplication of polynomials modulo an irreducible polynomial over $GF(2)$ of degree 8. For Rijndael, the inventors selected the following irreducible polynomial

$$m(x) = x^8 + x^4 + x^3 + x + 1 \text{ or } 11B.$$

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$$x^{13} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1 \; mod \; m(x)$$

$$= x^7 + x^6 + 1 = C1$$

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The End

Thanks a lot for your attention!