Shannon's Theory, Perfect Secrecy, and the One-Time Pad

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Outline

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Approaches to Evaluating the Security of a Cryptosystem

- **Computational security:** concerns the computational effort required to break a cryptosystem. A system to be **computationally secure** if the *best algorithm* for breaking it requires at least N operations,

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- **Computational security:** concerns the computational effort required to break a cryptosystem. A system to be **computationally secure** if the *best algorithm* for breaking it requires at least N operations, where $N = 2^{112}$.
- **Provable security:** is to provide evidence of security by means of a reduction. This approach only provides a proof of security relative to some other problem, not an absolute proof of security.
- **Unconditional security:** it cannot be broken, even with infinite computational resources.

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Perfect Secrecy

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Perfect Secrecy

- **Assumption:** The key K is chosen using some *fixed probability distribution* (often a key is chosen at random)
- The key is chosen before the sender knows what the plaintext P will be. Hence, we can assumption that the key and the plaintext are *independent random variables*.
- The two probability distributions on $\mathcal{P}$ and $\mathcal{K}$ induce a probability distribution on $\mathcal{C}$.
- $C(K)$ denotes the set of possible ciphertexts if K is the key. Then, for every $y \in \mathcal{C}$, we have that

$$\mathbf{Pr}[\mathbf{y} = y] = \sum_{\{K : y \in C(K)\}} \mathbf{Pr}[\mathbf{K} = K]\mathbf{Pr}[\mathbf{x} = d_K(y)].$$

Perfect Secrecy

- The conditional probability

$$\mathbf{Pr}[\mathbf{y} = y|\mathbf{x} = x] = \sum_{\{K:x=d_K(y)\}} \mathbf{Pr}[\mathbf{K} = K].$$

- The probability that x is the plaintext, given that y is the ciphertext

$$\mathbf{Pr}[\mathbf{x} = x|\mathbf{y} = y] = \frac{\mathbf{Pr}[\mathbf{x} = x] \times Pr[\mathbf{y} = y|\mathbf{x} = x]}{\mathbf{Pr}[\mathbf{y} = y]}$$

Definition

A cryptosystem has ***perfect secrecy*** *if*

$$\mathbf{Pr}[x|y] = \mathbf{Pr}[x] \ \forall x \in \mathcal{P}, y \in \mathcal{C}.$$

Perfect Secrecy

Theorem

Suppose $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$ *is a cryptosystem where* $|\mathcal{K}| = |\mathcal{C}| = |\mathcal{P}|$. *Then the cryptosystem provides* ***perfect secrecy*** *iff every key is used with equal probability* $\frac{1}{|\mathcal{K}|}$, *and for every* $x \in \mathcal{P}$ *and every* $y \in \mathcal{C}$, $\exists\, !\, K \;: e_K(x) = y$.

One-time Pad

Definition

Let $\mathcal{P} = \mathcal{C} = \mathcal{K} = (\mathbb{Z}_2)^n$ *for* $n \geq 1$. *For* $K \in (\mathbb{Z}_2)^n$ *, define* $e_K(x)$

$$e_K(x) = (x_1 + K_1, \ldots, x_n + K_n) \mod 2,$$

where $x = (x_1, \ldots, x_n)$ *and* $K = (K_1, \ldots, K_n)$.

Decryption is identical to encryption. If $y = (y_1, \ldots, y_n)$*, then*

$$d_K(y) = (y_1 + K_1, \ldots, y_n + K_n) \mod 2.$$

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Uncertainly and Information

- Tomorrow, the sun will rise from the East
- The phone will ring before the class is over.
- It will snow in Lucknow by the end of January

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Definition

*The **self information** of the event $X = x_i$ for $1 \leq i \leq n$ is defined as*

$$I(x_i) = \log\left(\frac{1}{P(x_i)}\right) = -\log(P(x_i))$$

Entropy

- Entropy can be thought of as a mathematical measure of information or uncertainty, and is computed as a function of a probability distribution.

Definition

Suppose $\mathbf{X}$ *is a discrete random variable. Then, the entropy or average self information of the random variable* $\mathbf{X}$ *is defined as*

$$H(\mathbf{X}) = -\sum_{x \in X} \mathbf{Pr}[x] \log_2 \mathbf{Pr}[x].$$

Properties of Entropy

Theorem

Suppose $\mathbf{X}$ *is a random variable having a probability distribution that takes on the values* $p_1, p_2, \ldots, p_n$,*where* $p_i > 0,\ 1 \leq i \leq n$. *Then* $H(\mathbf{X}) \leq \log_2 n$,

Properties of Entropy

Theorem

Suppose $\mathbf{X}$ *is a random variable having a probability distribution that takes on the values* $p_1, p_2, \ldots, p_n$*,where* $p_i > 0,\ 1 \leq i \leq n$*. Then* $H(\mathbf{X}) \leq \log_2 n$*,with equality iff* $p_i = 1/n,\ 1 \leq i \leq n$*.*

Theorem

$H(\mathbf{X}, \mathbf{Y}) \leq H(\mathbf{X}) + H(\mathbf{Y})$*, with equality if and only if* $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent random variables.*

Conditional Entropy

Definition

The conditional entropy $H(\mathbf{X}|\mathbf{Y})$ is defined by the weighted average over all possible values y. It is computed as

$$\begin{aligned} H(\mathbf{X}|\mathbf{Y}) &= \sum_y \mathbf{Pr}[y].H(\mathbf{X}|y) \\ &= -\sum_y \sum_x \mathbf{Pr}[y]\mathbf{Pr}[x|y] \log_2 \mathbf{Pr}[x|y]. \end{aligned}$$

Theorem

$$H(\mathbf{X}, \mathbf{Y}) = H(\mathbf{Y}) + H(\mathbf{X}|\mathbf{Y}).$$

Corollary

$H(\mathbf{X}|\mathbf{Y}) \leq H(\mathbf{X})$, *with equality iff $\mathbf{X}$ and $\mathbf{Y}$ are independent.*

Spurious Keys

Theorem

Let $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$ *be a cryptosystem. Then*

$$H(\mathbf{K}|\mathbf{C}) = H(\mathbf{K}) + H(\mathbf{P}) - H(\mathbf{C}).$$

Definition

- *Attacker to guess the key from the ciphertext shall guess the key and decrypt the cipher.*
- *He checks whether the plaintext obtained is 'meaningful' English. If not, he rules out the key.*
- *But due to the redundancy of language more than one key will pass this test.*
- *Those keys, apart from the correct key, are called* ***spurious****.*

Entropy of Plain Text

- H_L: measure of the amount of information per letter of '*meaningful*' strings of plaintext.
- A random string of plaintext formed using English letter has an entropy of $\log_2(26) \approx 4.76$ bits
- A first order entropy of the English text is $H(P) \approx 4.14$ bits
- A second order entropy of the English text is $\frac{H(P^2)}{2} \approx 3.56$ bits
- The entropy of a natural language L denoted by H_L and is defined by

$$H_L = \lim_{n\to\infty} \frac{H(P^n)}{n}$$

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Redundancy

Definition

The redundancy of L is defined as

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- For English Language, $1 \leq H_L \leq 1.5$. Let's take $H_L = 1.25$
- $|\mathcal{P}| = 26$
- $R_L = 0.75$

English Language is 75% redundant

Unicity Distance

Definition

The ***unicity distance*** *of a cryptosystem is defined to be the value of n, denoted by n_0, at which the expected number of spurious keys becomes zero i.e., the average amount of ciphertext required for an opponent to be able to uniquely compute the key, given enough computing time.*

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The End

Thanks a lot for your attention!